13-th Korean Mathematical Olympiad 2000

First Round – November 1999

1. Find all natural numbers x,y that satisfy $xy = 2^x - 1$.

2. Find all pairs of natural numbers (a,b) such that
$$\sqrt[3]{a+\sqrt{b}} + \sqrt[3]{a-\sqrt{b}} = 1.$$

3. In an acute triangle ABC with $\angle B \neq \angle C$, D is the foot of the altitude from A. The line AD meets the circumcircle of $\triangle ABC$ again at E. The foot of the perpendicular from E to BC is F. Prove that $S_{ADF} + S_{CEF} = S_{ABC}$, where S_{XYZ} denotes the area of $\triangle XYZ$.

4. An $l \times m \times n$ rectangular parallelepiped is made from unit cubes. How many of the cubes does the diagonal of the parallelepiped pass through?

5. A prime number p divides a^2+2 for a natural number a. Prove that p or $2p$ is of the form x^2+2y^2 for some natural numbers x,y.

6. Let $ABCD$ be a tetrahedron and K,L,M,N,P,Q be the midpoints of the edges AB,CD,AC,BD,AD,BC, respectively. Given that $AB = CD$, $AC = BD$, and $AD = BC$, prove that
$$\left(\frac{AB}{KL}\right)^2 + \left(\frac{AC}{MN}\right)^2 + \left(\frac{AD}{PQ}\right)^2 \geq 6.$$

7. The square $ACDE$ is drawn outside an equilateral triangle ABC. Let X be a point on the incircle of $ACDE$ and O be its center. Let Y be the circumcenter of $\triangle BCX$. Suppose the trace of point Y as X moves on the incircle is a segment PQ. Prove that $OP = OQ$.

8. Let be given two odd numbers a,b satisfying $b^2/4 < a < b^2/3$. Prove that there exist four integers x,y,z,w such that
$$\begin{aligned} x^2+y^2+z^2+w^2 &= a, \\ x+y+z+w &= a. \end{aligned}$$

   To solve this problem, you can use the fact that every positive integer n, not of the form $4^s(8t+7)$, can be represented as the sum of three squares.

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