Japanese Mathematical Olympiad 1996

Final Round – February 11

1. A plane is partitioned into triangles. Let $\mathscr{T}_0$ denote the set of vertices of triangles in the partition. Let ABC be a triangle with $A,B,C \in \mathscr{T}_0$ and θ be its smallest angle. Assume that no point of $\mathscr{T}_0$ lies inside the circumcircle of $\triangle ABC$. Prove that there exists a triangle σ in the partition such that its intersection with $\triangle ABC$ is nonempty and whose every angle is greater than θ.

   Remark. This problem is clearly wrong. We have it here so someone could potentially recognize the problem and tell us the correct one.

2. Let m,n be positive integers with $(m,n)=1$. Find $(5^m+7^m, 5^n+7^n)$.

3. Let $x>1$ be a real number which is not an integer. For each $n \in \mathbb{N}$, let $a_n = \left[x^{n+1}\right] - x\left[x^n\right]$. Prove that the sequence (a_n) is not periodic.

4. Let θ be the maximum of the six angles between six edges of a regular tetrahedron in space and a fixed plane. When the tetrahedron is rotated in space, find the maximum of θ.

5. Let q be a real number with $\dfrac{1+\sqrt{5}}{2} < q < 2$. If a positive integer n is represented in binary system as $n = 2^k + 2^{k-1}a_{k-1} + \cdots + 2a_1 + a_0$, where $a_i \in \{0,1\}$, define
$$p_n = q^k + q^{k-1}a_{k-1} + \cdots + qa_1 + a_0.$$
   Prove that there exist infinitely many positive integers m with the property that there is no $l \in \mathbb{N}$ such that $p_{2m} < p_l < p_{2m+1}$.