5-th Japanese Mathematical Olympiad 1995

Final Round – February 11

1. Let $n \geq 2$ and $r > 0$ be integers such that $n \nmid r$, and let g be the greatest common divisor of n and r. Prove that
$$\sum_{i=1}^{n-1} \left\langle \frac{ri}{n} \right\rangle = \frac{1}{2}(n-g),$$
where $\langle x \rangle = x - [x]$ is the fractional part of x.

2. Let $f(x)$ be a non-constant rational function (a quotient of two polynomials) and let a be a real number. Find all pairs (a, f) satisfying $f(x)^2 - a = f(x^2)$ for all x.

3. Let $ABCDE$ be a convex pentagon. Diagonal BE meets AC, AD at S, R, BD meets CA, CE at T, P, and CE meets AD at Q, respectively. Suppose the areas of triangles ASR, BTS, CPT, DQP, ERQ are all equal to 1.
   (a) Determine the area of pentagon $PQRST$.
   (b) Determine the area of pentagon $ABCDE$.

4. The sequence $(a_n)_{n\in\mathbb{N}}$ is defined by $a_{2n+1} = (-1)^n$ and $a_{2n} = a_n$. A point P moves on the coordinate plane as follows:
   (i) First P moves from the origin P_0 to $P_1(1,0)$
   (ii) After P has moved to P_i, it turns 90° to the left and moves forward 1 unit if $a_i = 1$, and turns 90° to the right and moves forward 1 unit if $a_i = -1$. Denote this point by P_{i+1}.

   Can the point retrace an edge? That is, can $P_u = P_w$ and $P_{u+1} = P_{w+1}$ for some u, w?

5. Let $1 \leq k \leq n$ be integers and $a_1, a_2, \dots, a_k$ be complex numbers which satisfy
$$a_1^i + a_2^i + \cdots + a_k^i = n \qquad \text{for } i = 1, 2, \dots, k.$$
Prove that $(x+a_1)(x+a_2)\cdots(x+a_k) = x^k + \binom{n}{1}x^{k-1} + \cdots + \binom{n}{k}$.

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