14-th Japanese Mathematical Olympiad 2004

Final Round – February 11

1. Prove that the is no positive integer n for which $2n^2+1$, $3n^2+1$ and $6n^2+1$ are all perfect squares.

2. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying
$$f(xf(x)+f(y)) = f(x)^2 + y \quad \text{for all } x, y \in \mathbb{R}.$$

3. Two orthogonal planes π_1 and π_2 are given in space. Let A and B be points on their intersection and C be a point on π_2 but not on π_1. The bisector of angle BCA meets AB at P. Denote by Σ the circle on π_1 with diameter AB. An arbitrary plane π_3 containing CP meets Σ at D and E. Prove that CP bisects $\angle DCE$.

4. For any positive numbers a, b, c with the sum 1 prove the inequality
$$\frac{1+a}{1-a} + \frac{1+b}{1-b} + \frac{1+c}{1-c} \leq 2 \leq \left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right).$$

5. In a certain country every town is connected by a road to exactly three other towns. A tourist traveling by roads visited each town exactly once and returned to the initial town. Next year he comes back for a round trip different from the last year's trip (i.e. not the same path in the reverse order), again visiting each town exactly once. Prove that he can always do so.

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