11-th Japanese Mathematical Olympiad 2001

Final Round – February 11

1. Each square of an $m \times n$ chessboard is painted black or white in such a way that for every black square, the number of black squares adjacent to it is odd (two squares are adjacent if they share one edge). Prove that the number of black squares is even.

2. An integer $n > 0$ is written in decimal system as $\overline{a_m a_{m-1} \dots a_1}$. Find all n such that
$$n = (a_m + 1)(a_{m-1} + 1) \cdots (a_1 + 1).$$

3. Three nonnegative real numbers a, b, c satisfy $a^2 \le b^2 + c^2$, $b^2 \le c^2 + a^2$ and $c^2 \le a^2 + b^2$. Prove the inequality
$$(a+b+c)(a^2+b^2+c^2)(a^3+b^3+c^3) \ge 4(a^6+b^6+c^6).$$
When does equality hold?

4. Let p be a prime number and m a positive integer. Show that there exists a positive integer n such that the decimal representation of p^n contains a string of m consecutive zeros.

5. Suppose that triangles ABC and PQR have the following properties:

   (i) A and P are the midpoints of QR and BC respectively,

   (ii) QR and BC are the bisectors of $\angle BAC$ and $\angle QPR$ respectively.

   Prove that $AB + AC = PQ + PR$.

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