Japanese Mathematical Olympiad 2000

Final Round

1. Consider the points $O(0,0)$ and $A(0,1/2)$ on the coordinate plane. Prove that there is no finite sequence of rational points $P_1,P_2,\dots,P_n$ in the plane such that

$$OP_1 = P_1P_2 = \cdots = P_{n-1}P_n = P_nA = 1.$$

2. Let $3n$ cards, denoted by distinct letters $a_1,a_2,\dots,a_{3n}$, be put in line in this order from left to right. After each shuffle, the sequence $a_1,a_2,\dots,a_{3n}$ is replaced by the sequence $a_3,a_6,\dots,a_{3n},a_2,a_5,\dots,a_{3n-1},a_1,a_4,\dots,\ a_{3n-2}$. Is it possible to replace the sequence of cards $1,2,\dots,192$ by the reverse sequence $192,191,\dots,1$ by a finite number of shuffles?

3. Given five points A,B,C,P,Q in a plane, no three of which are collinear, prove the inequality

$$AB+BC+CA+PQ \le AP+AQ+BP+BQ+CP+CQ.$$

4. Prove that for every natural number n there exists a set A_n with the following two properties:

   (i) A_n consists of n distinct natural numbers;

   (ii) for any $a \in A_n$, the remainder of the product of all elements of $A_n \setminus \{a\}$ divided by a is 1.

5. Finitely many lines are given in a plane. We call an *intersection point* a point that belongs to at least two of the given lines, and a *good intersection point* a point that belongs to exactly two lines. Assuming there are at least two intersection points, find the minimum number of good intersection points.

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