Italian IMO Team Selection Test 2007

First Day – Pisa, June 1

1. An acute triangle ABC is given.

   (a) Find the locus of the centers of rectangles with vertices on the sides of $\triangle ABC$;

   (b) Can a point be the center of three such rectangles?

2. On a tournament with $2n+1$ teams, every two teams play exactly one match and there are no draws. An unordered triple (A,B,C) of teams is called *cyclical* if A defeated B, B defeated C and C defeated A. Find the smallest and largest possible numbers of cyclical triples of teams in terms of n.

3. Find all functions $f:\mathbb{R}\to\mathbb{R}$ that satisfy

$$f(xy+f(x)) = xf(y)+f(x) \quad \text{for all } x,y.$$

Second Day – Pisa, June 2

4. The vertices and edges of a complete graph with n vertices are to be colored in such a way that (i) no two edges with a common vertex have the same color and (ii) no edge has the same color as either of its vertices. What smallest number of colors is necessary?

5. Let ABC be an acute triangle with sides a,b,c.

   (a) Find the locus of the points P such that the circumcenters O_a,O_b,O_c of the triangles PBC,PCA,PAB respectively satisfy

$$\frac{O_aO_b}{AB} = \frac{O_bO_c}{BC} = \frac{O_cO_a}{CA}.$$

   (b) For any such point P, show that the lines AO_a,BO_b,CO_c are concurrent at some point X.

   (c) Prove that the power of X with respect to the circumcircle of ABC equals $\frac{5R^2-a^2-b^2-c^2}{4}$.

6. Let $p \geq 5$ be a prime number.

   (a) Show that there exists a prime divisor $q \neq p$ of $N=(p-1)^p+1$.

   (b) If $\prod_{i=1}^{n} p_i^{a_i}$ is the canonical factorization of N, prove that

$$\sum_{i=1}^{n} a_i p_i \geq \frac{p^2}{2}.$$

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