15-th Italian Mathematical Olympiad 1999

Cesenatico, May 7, 1999

1. A rectangular sheet with sides a and b is fold along a diagonal. Compute the area of the overlapping triangle

2. We say that a natural number is *balanced* if the number of its decimal digits equals the number of its distinct prime factors. Prove that there are only finitely many balanced numbers.

3. Let r_1, r_2, r, with $r_1 < r_2 < r$, be the radii of three circles $\Gamma_1, \Gamma_2, \Gamma$, respectively. The circles Γ_1, Γ_2 are internally tangent to Γ at two distinct points A, B and intersect in two distinct points. Prove that the segment AB contains an intersection point of Γ_1 and Γ_2 if and only if $r_1 + r_2 = r$.

4. Albert and Barbara play the following game. On a table there are 1999 sticks, and each player in turn removes some of them: at least one stick, but at most half of the currently remaining sticks. The player who leaves just one stick on the table loses the game. Barbara moves first. Decide which player has a winning strategy and describe that strategy.

5. There is a village of pile-built dwellings on a lake, set on the gridpoints of an $m \times n$ rectangular grid. Each dwelling is connected by exactly p bridges to some of the neighboring dwellings (diagonal connections are not allowed; two dwellings can be connected by more than one bridge). Determine for which values m, n, p it is possible to place the bridges so that from any dwelling one can reach any other dwelling.

6. (a) Find all pairs (x, k) of positive integers such that $3^k - 1 = x^3$.

   (b) Prove that if $n > 1$ is an integer, $n \neq 3$, then there are no pairs (x, k) of positive integers such that $3^k - 1 = x^n$.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com