14-th Italian Mathematical Olympiad 1998

Cesenatico, May 8, 1998

1. Calculate the sum $\sum_{n=1}^{1\,000\,000}\left[\sqrt{n}\right]$. [You may use the formula $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$ without a proof.]

2. Prove that in each polyhedron there exist two faces with the same number of edges.

3. Alberto wants to organize a poker game with his friends this evening. Bruno and Barbara together go to gym once in three evenings, whereas Carla, Corrado, Dario and Davide are busy once in two evenings (not necessarily the same day). Moreover, Dario is not willing to play with Davide, since they have a quarrel over a girl.

   A poker game requires at least four persons (including Alberto). What is the probability that the game will be played?

4. Let $ABCD$ be a trapezoid with the longer base AB such that its diagonals AC an BD are perpendicular. Let O be the circumcenter of the triangle ABC and E be the intersection of the lines OB and CD. Prove that $BC^2 = CD \cdot CE$.

5. Suppose a_1, a_2, a_3, a_4 are distinct integers and $P(x)$ is a polynomial with integer coefficients satisfying $P(a_1) = P(a_2) = P(a_3) = P(a_4) = 1$.

   (a) Prove that there is no integer n such that $P(n) = 12$.

   (b) Do there exist such a polynomial and an integer n such that $P(n) = 1998$?

6. We say that a function $f : \mathbb{N} \to \mathbb{N}$ is *increasing* if $f(n) < f(m)$ whenever $n < m$, *multiplicative* if $f(nm) = f(n)f(m)$ whenever n and m are coprime, and *completely multiplcative* if $f(nm) = f(n)f(m)$ for all n, m.

   (a) Prove that if f is increasing then $f(n) \geq n$ for each n.

   (b) Prove that if f is increasing and completely multiplicative and $f(2) = 2$, then $f(n) = n$ for all n.

   (c) Does (b) remain true if the word ”completely” is omitted?

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