7-th Italian Mathematical Olympiad 1991

Cesenatico, May 1991

1. For every triangle ABC inscribed in a circle Γ, let A', B', C' be the intersections of the bisectors of the angles at A, B, C with Γ. Consider the triangle $A'B'C'$.

   (a) Do triangles $A'B'C'$ go over all possible triangles inscribed in Γ as $\triangle ABC$ varies? If not, what are the constraints?

   (b) Prove that the angle bisectors of $\triangle ABC$ are the altitudes of $\triangle A'B'C'$.

2. Prove that no number of the form $a^3 + 3a^2 + a$, for a positive integer a, is a perfect square.

3. We consider the sums of the form $\pm 1 \pm 4 \pm 9 \pm \cdots \pm n^2$. Show that every integer can be represented in this form for some n. (For example, $3 = -1 + 4$ and $8 = 1 - 4 - 9 + 16 + 25 - 36 - 49 + 64$.)

4. The squares of an 8×8 board are colored black and white in such a way that every row and every column contains exactly four black squares. Prove that the number of pairs of neighboring white squares is the same as the number of pairs of neighboring black squares. (Two squares are neighboring if they have a side in common.)

5. For which values of n does there exist a convex polyhedron with n edges?

6. We say that each positive number x has two *sons*: $x + 1$ and $\dfrac{x}{x+1}$. Characterize all the descendants of number 1.