23-rd Italian Mathematical Olympiad 2007

Cesenatico, May 11, 2007

1. Consider a regular hexagon in a plane. For each point P in the plane, denote by $l(P)$ the sum of distances of P from the lines containing the sides of the hexagon, and by $v(P)$ the sum of distances of P from the vertices of the hexagon.
   (a) For which points P is $l(P)$ minimal?
   (b) For which points P is $v(P)$ minimal?
2. We say that polynomials p and q with integer coefficients are *similar* if they have the same degree and the coefficients which differ only in order.
   (a) Prove that if p and q are similar then $p(2007)-q(2007)$ is even.
   (b) Is there an integer $k>2$ such that $p(2007)-q(2007)$ is a multiple of k for any two similar polynomials p and q?
3. Let G be the centroid of a triangle ABC, $D \neq A$ be the point on ray AG with $AG=GD$, $E \neq B$ be the point on ray BG with $BG=GE$, and M be the midpoint of AB. Show that the quadrilateral $BMCD$ is cyclic if and only if $BA=BE$.
4. Having lost a bet to Barbara, Alberto proposes the following game. Starting with the numbers $0,1,\dots,1024$, Barbara deletes 2^9 numbers on her choice; then Alberto deletes 2^8 of the remaining numbers, then Barbara removes 2^7 numbers etc. At the end two numbers a and b remain. Then Alberto pays Barbara $|a-b|$ euros. What largest amount of money can Barbara earn independent of Alberto's strategy?
5. Consider the sequence given by $x_1=2$, $x_{n+1}=2x_n^2-1$ for $n \geq 1$. Prove that n and x_n are coprime for each $n \geq 1$.
6. Let $n \geq 2$ be a given integer. Determine
   (a) the largest real c_n such that $\frac{1}{1+a_1}+\frac{1}{1+a_2}+\cdots+\frac{1}{1+a_n} \geq c_n$ holds for any positive numbers $a_1,\dots,a_n$ with $a_1a_2\cdots a_n=1$.
   (b) the largest real d_n such that $\frac{1}{1+2a_1}+\frac{1}{1+2a_2}+\cdots+\frac{1}{1+2a_n} \geq d_n$ holds for any positive numbers $a_1,\dots,a_n$ with $a_1a_2\cdots a_n=1$.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com