15-th Iranian Mathematical Olympiad 1997/1998

Third Round

Time: 4 hours each day.

First Day

1. Let $f_1, f_2, f_3 : \mathbb{R} \to \mathbb{R}$ be functions such that $a_1 f_1 + a_2 f_2 + a_3 f_3$ is monotonous for all $a_1, a_2, a_3 \in \mathbb{R}$. Show that there exist real numbers c_1, c_2, c_3, not all zero, such that
$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0 \quad \text{for all real } x.$$

2. Let X be a set of n elements, and $A_1, A_2, \dots, A_m$ be three-element subsets of X such that for all pairs of distinct indices i, j, $|A_i \cap A_j| \le 1$. Prove that there exists a subset A of X with at least $\left[\sqrt{2n}\right]$ elements containing none of the subsets A_i.

3. The edges of a regular 2^n-gon are colored red and blue in some fashion. A step consists in recoloring each edge whose neighbors are both the same color in red, and recoloring each edge whose neighbors are of opposite colors in blue. Prove that after 2^{n-1} steps all of the edges will be red, and show that this needn't hold after fewer steps.

Second Day

4. Let $n_1 < n_2 < \cdots$ be a sequence of natural numbers such that for $i < j$ the decimal representation of n_i does not occur as the leftmost digits of the decimal representation of n_j. (For example, 137 and 13729 cannot both occur in the sequence.) Prove that
$$\sum_i \frac{1}{n_i} \le 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{9}.$$

5. Let ABC be a triangle and D be the point on the extension of side BC past C such that $CD = AC$. The circumcircle of ACD intersects the circle with diameter BC again at P. Let BP meet AC at E and CP meet AB at F. Prove that the points D, E, F are collinear.

6. Let $\mathscr{K}$ be a convex polygon in the plane. Show that for any triangle of the minimum possible area containing $\mathscr{K}$, the midpoints of its sides lie on $\mathscr{K}$.

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The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
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