24-th Iranian Mathematical Olympiad 3006/2007

Third Round

Time: 4 hours each day.

First Day

1. Let A be the largest subset of $\{1,\dots,n\}$ such that each element of A divides the most one other element of A. Prove that
$$\frac{2n}{3} \le |A| \le 3\left\lceil \frac{n}{4} \right\rceil.$$

2. Does there exist a sequence of positive integers $a_0, a_1, a_2, \dots$ such that for each $i \neq j$, $(a_i, a_j) = 1$ and for all n, the polynomial $\sum_{i=0}^{n} a_i x^i$ is irreducible in $\mathbb{Z}[x]$?

3. A triangle ABC is isosceles ($AB = AC$). From A, we draw a line l parallel to BC. P and Q are on the perpendicular bisectors of AB and AC respectively such that $PQ \perp BC$. M and N are points on l such that $\angle APM = \angle AQN = 90^\circ$. Prove that
$$\frac{1}{AM} + \frac{1}{AN} \le \frac{2}{AB}.$$

Second Day

4. Suppose that n are placed in a plane, such that no two are parallel and no three are concurrent. For each two of the lines let the angle between them be the smallest angle produced at their intersection. Find the largest value of the sum of the $\binom{n}{2}$ angles between the lines.

5. O is a point inside the triangle ABC such that $OA = OB + OC$. Let B', C' be the midpoints of the arcs AOC and AOB respectively. Prove that the circumcircles COC' and BOB' are tangent to each other.

6. Find all polynomials of degree 3, such that for each nonnegative reals x and y:
$$p(x+y) \ge p(x) + p(y).$$

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