21-st Iranian Mathematical Olympiad 2003/04

Third Round

1. Let $ABCD$ be a cyclic quadrilateral. The perpendiculars to AD and BC at A and C respectively meet at M, and the perpendiculars to AD and BC at D and B meet at N. If the lines AD and BC meet at E, prove that $\angle DEN = \angle CEM$.

2. A finite alphabet S is given. A word is a finite sequence of letters from S. We are given m forbidden words. Suppose that there exists an infinite sequence of letters from S that contains none of the forbidden words as a block. Show that there exists a sequence of letters that is infinite on both sides and contains none of the forbidden words as a block.

3. Let A be a finite set of prime numbers, and let a be a positive integer. Prove that there are only finitely many positive integers m for which all prime divisors of $a^m - 1$ are in A.

4. Let M and M' be isogonally conjugate points in a triangle ABC. Let P, Q, R be the orthogonal projections of M on BC, CA, AB respectively, and let P', Q', R' be the corresponding projections of M'. The lines QR and $Q'R'$, RP and $R'P'$, PQ and $P'Q'$ intersect at E, F, G respectively. Prove that the lines EA, FB, and GC are parallel.

5. If p is a prime number of the form $4k + 1$ ($k \in \mathbb{N}$), prove that the equation $x^2 - py^2 = -1$ has a solution in integers.

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