19-th Iranian Mathematical Olympiad 2001/02

Third Round

1. Find all real polynomials $P(x)$ such that $P(a) \in \mathbb{Z}$ implies that $a \in \mathbb{Z}$.

2. Let B_1 be an arbitrary point outside a fixed ellipse $\mathcal{E}$. If a tangent from B_1 to $\mathcal{E}$ touches E at point C_1, denote by B_2 the point symmetric to B_1 with respect to C_1. For each integer $i \geq 2$, define B_{i+1} in terms of B_i in this manner so that $b_{i+1} \neq B_{i-1}$. Prove that the sequence (B_i) is bounded in the plane.

3. In a triangle ABC, define $\mathcal{C}_a$ to be the circle tangent to sides AB and AC and internally to the circumcircle of $\triangle ABC$, and denote by r_a the radius of $\mathcal{C}_a$. Define r_b and r_c analogously. If r is the inradius of $\triangle ABC$, prove that
$$r_a + r_b + r_c \geq 4r.$$

4. Let n and k be integers with $2 \leq k \leq n$. Let $\mathcal{F}$ be a subset of $\mathbb{P}(\{1, \dots, n\})$ with the property that, for every $F, G \in \mathcal{F}$, there exists an integer t with $1 \leq t \leq n$ such that $\{t, t+1, \dots, t+k-1\} \subseteq F \cap G$. Prove that $|\mathcal{F}| \leq 2^{n-k}$.

5. For every real number x define $\langle x \rangle = \min\{\{x\}, 1-\{x\}\}$, where $\{x\}$ denotes the fractional part of x. Prove that, for every irrational number α and every positive real number ε, there exists a positive integer n such that $\langle n^2\alpha \rangle < \varepsilon$.

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