23-rd Iranian Mathematical Olympiad 2005/06

Second Round

Time: 4 hours each day.

First Day

1. Let $P(x)$ be an irreducible polynomial of an odd degree with rational coefficients. Rational polynomials $Q(x)$ and $R(x)$ are such that $P(x)$ divides $Q(x)^2 + Q(x)R(x) + R(x)^2$. Prove that $P(x)^2$ also divides $Q(x)^2 + Q(x)R(x) + R(x)^2$.

2. In a triangle ABC, H is the orthocenter, O the circumcenter and ω the circumcircle. Line AO meets ω again at A_1, and A_1H and AH intersect ω again at A' and A'' respectively. Points B', B'', C', C'' are similarly defined. Prove that the lines $A'A'', B'B''$ and $C'C''$ are concurrent at a point on the Euler line of $\triangle ABC$.

3. If nonnegative real numbers a, b, c satisfy $\frac{1}{a^2+1} + \frac{1}{b^2+1} + \frac{1}{c^2+1} = 2$, prove that $ab + bc + ca \leq \frac{3}{2}$.

Second Day

4. Let k be an integer. The sequence $(a_n)_{n=1}^{\infty}$ is defined by

$$a_0 = 0,\ a_1 = 1;\ \ a_n = 2ka_{n-1} - (k^2+1)a_{n-2} \quad \text{for } n \geq 2.$$

   If p is a prime number of the form $4m + 3$, prove that

   (a) $a_{n+p^2-1} \equiv a_n \pmod{p}$;

   (b) $a_{n+p^3-p} \equiv a_n \pmod{p^2}$.

5. The 27-element sets $A_1, A_2, \dots, A_{35}$ have the property that every three of them have exactly one element in common. Show that all the 35 sets A_i have an element in common.

6. In a triangle ABC, point L is taken on side BC and points M, N on the extensions of AB and AC over B and C, respectively, such that $\angle ALC = 2\angle AMC$ and $\angle ALB = 2\angle ANB$. Let O be the circumcenter of triangle AMN. Show that OL is perpendicular to BC.