17-th Iranian Mathematical Olympiad 1999/2000

Second Round

Time: 4 hours each day.

First Day

1. Two circles intersect at points A and B. A line l through A meets these circles again at C and D. Let M and N be the midpoints of the arcs BC and BD not containing A, and K be the midpoint of CD. Show that $\angle MKN = 90^\circ$.

2. Let A and B be arbitrary finite sets and let $f : A \to B$ and $g : B \to A$ be functions such that g is not onto. Prove that there is a subset S of A such that
$$A \setminus S = g(B \setminus f(S)).$$

3. Suppose $f : \mathbb{N} \to \mathbb{N}$ is a function that satisfies $f(1) = 1$ and
$$f(n+1) = \begin{cases} f(n) + 2 & \text{if } n = f(f(n) - n + 1), \\ f(n) + 1 & \text{otherwise.} \end{cases}$$
   (a) Prove that $f(f(n) - n + 1)$ is either n or $n + 1$.
   (b) Determine f.

Second Day

4. Let us denote $\Pi = \{(x, y) \mid y > 0\}$. We call a semicircle in Π with center on the x-axis a *semi-line*. Two intersecting semi-lines determine four *semi-angles*. A *bisector* of a semi-angle is a semi-line that bisects the semi-angle. Prove that in every semi-triangle (determined by three semi-lines) the bisectors are concurrent.

5. Determine all functions $f : \mathbb{N} \to \mathbb{N}$ such that:
   (i) $f(m) = 1$ if and only if $m = 1$;
   (ii) If $d = (m, n)$, then $f(mn) = \dfrac{f(m)f(n)}{f(d)}$;
   (iii) $f^{2000}(m) = m$ for every $m \in \mathbb{N}$.

6. Let n points be given on a circle and let $nk + 1$ chords between these points be drawn, where $2k + 1 < n$. Show that it is possible to select $k + 1$ of the chords so that no two of them intersect.

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