9-th Irish Mathematical Olympiad 1996

May 4, 1995

Time: 3 hours each part.

Part 1

1. For each positive integer n, let $f(n)$ denote the greatest common divisor of $n!+1$ and $(n+1)!$. Find, with proof, a formula for $f(n)$.

2. Let $S(n)$ denote the sum of the digits of a natural number n (in base 10). Prove that for every n,
$$S(2n) \le 2S(n) \le 10S(2n).$$
Prove also that there is a positive integer n with $S(n) = 1996S(3n)$.

3. A function f from $[0,1]$ to $\mathbb{R}$ has the following properties:
   (i) $f(1) = 1$;
   (ii) $f(x) \ge 0$ for all $x \in [0,1]$;
   (iii) If $x, y, X + y \in [0,1]$, then $f(x+y) \ge f(x) + f(y)$.

   Prove that $f(x) \le 2x$ for all $x \in [0,1]$.

4. Let F be the midpoint of the side BC of a triangle ABC. Isosceles right-angled triangles ABD and ACE are constructed externally on AB and AC with the right angles at D and E. Prove that the triangle DEF is right-angled and isosceles.

5. Show how to dissect a square into at most five pieces in such a way that the pieces can be reassembled to form three squares of (pairwise) distinct areas.

Part 2

6. The Fibonacci sequence is defined by $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_n + F_{n+1}$ for $n \ge 0$. Prove that:
   (a) The statement "$F_{n+k} - F_n$ is divisible by 10 for all $n \in \mathbb{N}$" is true if $k = 60$ but false for any positive integer $k < 60$.
   (b) The statement "$F_{n+t} - F_n$ is divisible by 100 for all $n \in \mathbb{N}$" is true if $t = 300$ but false for any positive integer $k < 300$.

7. Show that for every positive integer n,
$$2^{1/2} \cdot 4^{1/4} \cdot 8^{1/8} \cdots (2^n)^{1/2^n} < 4.$$

8. Suppose that p is a prime number and a and n positive integers such that $2^p + 3^p = a^n$. Prove that $n = 1$.

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9. In an acute-angled triangle ABC, D, E, F are the feet of the altitudes from A, B, C, respectively, and P, Q, R are the feet of the perpendiculars from A, B, C onto the EF, FD, DE, respectively. Prove that the lines AP, BQ, CR are concurrent.

10. The following game is played on a rectangular chessboard 5×9 (with five rows and nine columns). Initially, a number of discs are randomly placed on some of the squares of the chessboard, with at most one disc on each square. A complete move consists of the moving every disc subject to the following rules:

    (i) Each disc may be moved one square up, down, left or right;

    (ii) If a particular disc is moved up or down as part of a complete move, then it must be moved left or right in the next complete move;

    (iii) If a particular disc is moved left or right as part of a complete move, then it must be moved up or down in the next complete move;

    (iv) At the end of a complete move, no two discs can be on the same square.

    The game stops if it becomes impossible to perform a complete move. Prove that if initially 33 discs are placed on the board then the game must eventually stop. Prove also that it is possible to place 32 discs on the boards in such a way that the game could go on forever.