8-th Irish Mathematical Olympiad 1995

May 6, 1995

Time: 3 hours each part.

Part 1

1. There are n^2 students in a class. Each week all the students participate in a table quiz. Their teacher arranges them into n teams of n players each. For as many weeks as possible, this arrangement is done in such a way that any pair of students who were members of the same team one week are not in the same team in subsequent weeks. Prove that after at most $n+2$ weeks, it is necessary for some pair of students to have been members of the same team in at least two different weeks.

2. Determine all integers a for which the equation $x^2+axy+y^2=1$ has infinitely many distinct integer solutions x,y.

3. Points A,X,D lie on a line in this order, point B is on the plane such that $\angle ABX > 120^\circ$, and point C is on the segment BX. Prove the inequality
$$2AD \geq \sqrt{3}(AB+BC+CD).$$

4. Consider the following one-person game played on the real line. During the game disks are piled at some of the integer points on the line. To perform a move in the game, the player chooses a point j at which at least two disks are piled and then takes two disks from the point j and places one of them at $j-1$ and one at $j+1$. Initially, $2n+1$ disks are placed at point 0. The player proceeds to perform moves as long as possible. Prove that after $\frac{1}{6}n(n+1)(2n+1)$ moves no further moves will be possible and that at this stage, one disks remains at each of the positions $-n,-n+1,\dots,0,\dots,n$.

5. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that for all real numbers x,y
$$xf(x)-yf(y)=(x-y)f(x+y).$$

Part 2

6. Prove that for every positive integer n,
$$n^n \leq (n!)^2 \leq \left(\frac{(n+1)(n+2)}{6}\right)^n.$$

7. Let a,b,c be complex numbers. Prove that if all the roots of the equation $x^3+ax^2+bx+c=0$ are of module 1, then so are the roots of the equation $x^3+|a|x^2+|b|x+|c|=0$.

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8. Denote $S = [0,1] \times [0,1]$. For each real number t with $0 < t < 1$, let C_t be the set of all points $(x,y) \in S$ that are on or above the line joining $(t,0)$ to $(0,1-t)$. Prove that the points common to all C_t are those points in S which are on or above the curve $\sqrt{x} + \sqrt{y} = 1$.

9. Points P, Q, R are given in the plane. It is known that there is a triangle ABC such that P is the midpoint of BC, Q the point on side CA with $CQ/QA = 2$, and R the point on side AB with $AR/RB = 2$. Determine with proof how the triangle ABC may be reconstructed from P, Q, R.

10. For each integer n of the form $n = p_1 p_2 p_3 p_4$, where p_1, p_2, p_3, p_4 are distinct primes, let $1 = d_1 < d_2 < \cdots < d_{15} < d_{16} = n$ be the divisors of n. Prove that if $n < 1995$, then $d_9 - d_8 \neq 22$.

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