5-th Irish Mathematical Olympiad1992

May 02, 1992

Time: 3 hours each part.

Part 1

1. Describe in geometric terms the set of points (x,y) in the plane such that x and y satisfy the condition $t^2+yt+x \geq 0$ for all t with $-1 \leq t \leq 1$.

2. How many ordered triples (x,y,z) of real numbers satisfy the system of equations
$$\begin{aligned} x^2+y^2+z^2 &= 9, \\ x^4+y^4+z^4 &= 33, \\ xyz &= -4? \end{aligned}$$

3. Let A be a nonempty set with n elements. Find the number of ways of choosing a pair of subsets (B,C) of A such that B is a nonempty subset of C.

4. In a triangle ABC, the points $A?$, $B?$, and $C?$ on the sides opposite A, B, and C, respectively, are such that the lines $AA?$, $BB?$, and $CC?$ are concurrent. Prove that the diameter of the circumscribed circle of the triangle ABC equals the product $AB? \cdot BC? \cdot CA?$ divided by the area of the triangle $A?B?C?$.

5. Let ABC be a triangle such that the coordinates of the points A and B are rational numbers. Prove that the coordinates of C are rational if, and only if,$\tan A$, $\tan B$, and $\tan C$, when dened, are all rational numbers.

Part 2

6. Let $n > 2$ be an integer and let $m = \sum k^3$, where the sum is taken over all integers k with $1 \leq k < n$ that are relatively prime to n. Prove that $n|m$.

7. If a_1 is a positive integer, form the sequence $a_1, a_2, a_3, \ldots$ by letting a_2 be the product of the digits of a_1 , etc.. If a_k consists of a single digit, for some $k \geq 1$, a_k is called a *digital root* of a_1. It is easy to check that every positive integer has a unique digital root. (For example, if $a_1 = 24378$, then $a_2 = 1344$, $a_3 = 48$, $a_4 = 3_2$, $a_5 = 6$, and thus 6 is the digital root of 24378.) Prove that the digital root of a positive integer n equals 1 if, and only if, all the digits of n equal 1.

8. Let a, b, c, and d be real numbers with $a \neq 0$. Prove that if all the roots of the cubic equation
$$az^3+bz^2+cz+d = 0$$
lie to the left of the imaginary axis in the complex plane, then
$$ab > 0, bc-ad > 0, ad > 0.$$

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9. A convex pentagon has the property that each of its diagonals cuts o? a triangle of unit area. Find the area of the pentagon.

10. If, for $k = 1, 2, \dots, n$, a_k and b_k are positive real numbers, prove that

$$\sqrt[n]{a_1 a_2 \cdots a_n} + \sqrt[n]{b_1 b_2 \cdots b_n} \leq \sqrt[n]{(a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n)};$$

and the equality holds if and only if

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}.$$

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