20-th Irish Mathematical Olympiad 2007

May 12, 2007

Time: 3 hours each part.

Part 1

1. Find all prime numbers p and q such that p divides $q+6$ and q divides $p+7$.

2. Prove that a triangle ABC is right-angled if and only if
$$\sin^2 A + \sin^2 B + \sin^2 C = 2.$$

3. The point P is a fixed point on a circle and Q is a fixed point on a line. The point R is a variable point on the circle such that P, Q, and R are not collinear. The circle through P, Q, and R meets the line again at V. Show that the line VR passes through a fixed point.

4. Air Michael and Air Patrick operate direct flights connecting Belfast, Cork, Dublin, Galway, Limerick, and Waterord. For each pair of cities exactly one of the airlines operates the route (in both directions) connecting the cities. Prove that there are four cities for which one of the airlines operates a round trip. (Note that a round trip of four cities P, Q, R, and S, is a journey that follows the path $P \to Q \to R \to S \to P$.)

5. Let r and n be nonnegative integers such that $r \le n$.

   (a) Prove that
   $$\frac{n+1-2r}{n+1-r}\binom{n}{r}$$
   is an integer.

   (b) Prove that
   $$\sum_{r=0}^{[n/2]} \frac{n+1-2r}{n+1-r}\binom{n}{r} < 2^{n-2}$$
   for all $n \ge 9$.

Part 2

6. Let r, s, and t be the roots of the cubic polynomial
$$p(x) = x^3 - 2007x + 2002.$$
Determine the value of
$$\frac{r-1}{r+1} + \frac{s-1}{s+1} + \frac{t-1}{t+1}.$$

7. Suppose that a, b, and c are positive real numbers. Prove that

$$\frac{a+b+c}{3} \leq \sqrt{\frac{a^2+b^2+c^2}{3}} \leq \frac{\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}}{3}.$$

For each of the inequalities, find the conditions on a, b, and c such that equality holds.

8. Let ABC be a triangle the lengths of whose sides BC, CA, AB, respectively, are denoted by a, b, and c. Let the internal bisectors of the angles $\angle BAC$, $\angle ABC$, $\angle BCA$, respectively, meet the sides BC, CA, and AB at D, E, and F. Denote the lengths of the line segments AD, BE, CF by d, e, and f, respectively. Prove that

$$def = \frac{4abc(a+b+c)\Delta}{(a+b)(b+c)(c+a)},$$

where Δ stands for the area of the triangle ABC.

9. Find the number of zeros in which the decimal expansion of 2007! ends. Also find its last non-zero digit.

10. Suppose that a and b are real numbers such that the quadratic polynomial $f(x) = x^2+ax+b$ has no nonnegative real roots. Prove that there exist two polynomials g, h whose coefficients are nonnegative real numbers such that

$$f(x) = \frac{g(x)}{h(x)}$$

for all real numbers x.

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