17-th Irish Mathematical Olympiad 2004

May 08, 2004

Time: 3 hours each part.

Part 1

1. (a) For which positive integers n, does $2n$ divide the sum of the first n positive integers?

   (b) Determine with proof those positive integers n (if any) which have the property that $2n+1$ divides the sum of the first n positive integers.

2. Each of the players in a tennis tournament played one match against each of the others. If every player won at least one match, show that there is a group A, B, C of three players for which A beat B, B beat C, and C beat A.

3. AB is a chord of length 6 of a circle centered at O and of radius 5. Let $PQRS$ denote the square inscribed in the sector OAB such that P is on the radius OA, S is on the radius OB and Q and R are points on the arc of the circle between A and B. Find the area of $PQRS$.

4. Prove that there are only two real numbers x such that

$$(x-1)(x-2)(x-3)(x-4)(x-5)(x-6)=720.$$

5. Let $a,b\geq 0$. Prove that

$$\sqrt{2}\left(\sqrt{a(a+b)^3}+b\sqrt{a^2+b^2}\right)\leq 3(a^2+b^2),$$

   with equality if and only if $a=b$.

Part 2

6. Determine all pairs of prime numbers (p,q), with $2\leq p,q<100$ such that $p+6$, $p+10$, $q+4$, $q+10$, and $p+q+1$ are all prime numbers.

7. A and B are distinct points on a circle T. C is a point distinct from B such that $|AB|=|AC|$, and such that BC is tangent to G at B. Suppose that the bisector of $\angle ABC$ meets AC at a point D inside T. Show that $\angle ABC>72^\circ$.

8. Suppose n is an integer ≥ 2. Determine the first digit after the decimal point in the decimal expansion of the number

$$\sqrt[3]{n^3+2n^2+n}.$$

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9. Define the function m of three real variables x, y, z by

$$m(x,y,z) = \max\{x^2, y^2, z^2\}, \ \ x,y,z \in \mathbb{R}.$$

Determine, with proof, the minimum value of m if x, y, z vary in $\mathbb{R}$ subject to the following restrictions:

$$x+y+z=0, \ \ x^2+y^2+z^2=1.$$

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