15-th Irish Mathematical Olympiad 2002

May 11, 2002

Time: 3 hours each part.

Part 1

1. In triangle ABC with $AB = 20$, $AC = 21$ and $BC = 29$, points D and E are taken on the segment BC such that $BD = 8$ and $EC = 9$. Calculate the angle $\angle DAE$.

2. (a) A group of people attends a party. Each person has at most three acquaintances in the group, and if two people do not know each other, then they have a common acquaintance in the group. What is the maximum possible number of people present?

   (b) If, in addition, the group contains three mutual acquaintances, what is the maximum possible number of people?

3. Find all triples of positive integers (p, q, n), with p and q primes, satisfying
$$p(p+3)+q(q+3) = n(n+3).$$

4. The sequence (a_n) is defined by $a_1 = a_2 = a_3 = 1$ and
$$a_{n+1}a_{n-2} - a_n a_{n-1} = 2 \quad \text{for all } n \geq 3.$$
Prove that a_n is a positive integer for all $n \geq 1$.

5. Let $0 < a, b, c < 1$. Prove the inequality
$$\frac{a}{1-a} + \frac{b}{1-b} + \frac{c}{1-c} \geq \frac{3\sqrt[3]{abc}}{1-\sqrt[3]{abc}}.$$
Determine the cases of equality.

Part 2

6. A $3 \times n$ grid is filled as follows. The first row consists of the numbers from 1 to n arranged in ascending order. The second row is a cyclic shift of the top row: $i, i+1, \dots, n, 1, 2, \dots, i-1$ for some i. The third row has the numbers 1 to n in some order so that in each of the n columns, the sum of the three numbers is the same. For which values of n is it possible to fill the grid in this way? For all such n, determine the number of different ways of filling the grid.

7. Suppose n is a product of four distinct primes a, b, c, d such that

   (i) $a + c = d$;

   (ii) $a(a+b+c+d) = c(d-b)$;

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(iii) $1+bc+d=bd$.

Determine n.

8. Find all functions $f:\mathbb{Q}\to\mathbb{Q}$ such that

$$f(x+f(y))=y+f(x) \quad \text{for all } x,y\in\mathbb{Q}.$$

9. Let $\alpha=2+\sqrt{3}$. Prove that $\alpha^n-[\alpha^n]=1-\alpha^{-n}$ for all $n\in\mathbb{N}_0$.

10. Let ABC be a triangle with integer side lengths, and let its incircle touch BC at D and AC at E. If $|AD^2-BE^2|\le 2$, show that $AC=BC$.