# Indian IMO Team Selection Tests 1998

## First Practice Test

Mumbai, May 9

1. Points $D, E, F$ are taken on the sides $BC, CA, AB$ of a triangle $ABC$ so that $BD = CE = AF$. Show that if the circumcenters of triangles $ABC$ and $DEF$ coincide then $\triangle ABC$ is equilateral.

2. A triangle $PQR$ in the coordinate plane has the vertices at lattice points and sides of integer lengths. If $PQ = PR$, prove that the midpoint $M$ of $QR$ is a lattice point and that the length of the altitude $PM$ is an integer.

3. Prove that there is no function $f : \mathbb{Z} \to \mathbb{Z}$ such that for all $m, n \in \mathbb{Z}$
$$f(m) = n \quad \text{implies} \quad f(n) = m \text{ and } f(n+3) = 3m.$$

4. For each permutation $\sigma = (a_1, a_2, \dots, a_n)$ of numbers $1, 2, \dots, n$ define
$$S(\sigma) = \frac{a_1}{1} + \frac{a_2}{2} + \cdots + \frac{a_n}{n}.$$
Find positive integers $n$ and $N$ for which all the integers $m$ with $N \le m \le N + 100$ occur among the values taken by $S(\sigma)$ as $\sigma$ varies over all permutations of $1, \dots, n$.

## Second Practice Test

Mumbai, May 14

1. Circle $\mathscr{S}_1$ and $\mathscr{S}_2$ touch each other externally at $P$ and both touch a circle $\mathscr{S}$ internally at $B$ and $C$ respectively. The common tangent to $\mathscr{S}_1$ and $\mathscr{S}_2$ at $P$ meets $\mathscr{S}$ at two points, one of which is denoted by $A$. Lines $AB$ and $AC$ meet $\mathscr{S}_1$ and $\mathscr{S}_2$ again at $Q$ and $R$ respectively. Show that $QR$ is a common tangent to $\mathscr{S}_1$ and $\mathscr{S}_2$.

2. (a) Prove that the product of two numbers of the form $a^2 + 3b^2$ ($a, b \in \mathbb{Z}$) is again of the same form.
   (b) If $7n$ is of the above form and $n$ is an integer, show that $n$ itself is of that form.

3. Given three nonnegative numbers $a, b, c$, define
$$X = a + b + c + 2\sqrt{a^2 + b^2 + c^2 - bc - ca - ab}.$$
   (a) Prove that $X \ge \max\{3a, 3b, 3c\}$.
   (b) One of the numbers $\sqrt{X - 3a}, \sqrt{X - 3b}, \sqrt{X - 3c}$ equals the sum of the other two.

4. Let $X$ be an $(n+2)$-element subset of $X = \{-n, -n+1, \dots, n\}$, where $n$ is a given positive integer. Show that there exist distinct numbers $a, b, c \in X$ such that $a+b=c$.

## First Test

Mumbai, May 17

1. The altitudes $AK$, $BL$, $CM$ in the triangle $ABC$ meet at $H$. Let $P$ be the midpoint of $AH$. If $BH$ and $MK$ intersect at $S$ and $LP$ and $AM$ intersect at $T$, show that $TS \perp BC$.

2. Find the number of integers $x$ with $|x| \le 1997$ for which 1997 divides $x^2+(x+1)^2$.

3. If $x, y, z$ are nonegative real numbers such that $xy+yz+zx+xyz=4$, prove that
$$x+y+z \ge xy+yz+zx.$$

4. Let $a_1 \ge a_2 \ge \cdots \ge a_n$ be integer powers of 2. If $a_1 \le 2000$ and $a_1+a_2+\cdots+a_n \ge 2048$, show that $a_1+a_2+\cdots+a_m = 2048$ holds for some $m$.

## Second Test

Mumbai, May 20

1. A circle lying inside a parallelogram $ABCD$ touches the sides $AB$ and $AD$ and intersects $BD$ at $E$ and $F$. Show that there exists a circle passing through $E$ and $F$ and touching the sides $CB$ and $CD$, possibly extended.

2. Find all triples $(x, y, n)$ of positive integers with $(x, n+1) = 1$ such that $x^n+1 = y^{n+1}$.

3. Real numbers $x, y, z$ satisfy $x^2 = y+2$, $y^2 = z+2$, $z^2 = x+2$. If $xyz$ is negative, find all possible values of $x^2+y^2+z^2$.

4. In town $X$, there are $n$ girls and $n$ boys, and each girl knows each boy. In town $Y$, there are $n$ girls $g_1, g_2, \dots, g_n$ and $2n-1$ boys $b_1, b_2, \dots, b_{2n-1}$. The girl $g_i$, $i = 1, 2, \dots, n$, knows the boys $b_1, b_2, \dots, b_{2i-1}$, and no others. For all $r = 1, 2, \dots, n$, denote by $X(r), Y(r)$ the number of different ways in which $r$ girls from town $X$, respectively town $Y$, can dance with $r$ boys from their own town, forming $r$ pairs, each girl with a boy she knows. Prove that $X(r) = Y(r)$ for each $r = 1, 2, \dots, n$.

## Third Test

Mumbai, May 23

1. Let $A_0B_0C_0$ be a triangle with incenter $I$. For $n \geq 1$ form the triangles $A_nB_nC_n$ succesively, where $A_n, B_n, C_n$ are the midpoints of $B_{n-1}C_{n-1}$, $C_{n-1}A_{n-1}$, $A_{n-1}B_{n-1}$. Let $r(A_0B_0C_0)$ be the largest integer $n$ for which $I$ is in the interior of $\triangle A_nB_nC_n$.

   (a) Show that $r(A_0B_0C_0) \geq 1$;

   (b) Find the maximum value of $r(A_0B_0C_0)$ among all integer sided triangles $A_0B_0C_0$ with perimeter 98.

2. Given a positive integer $m$, consider the set $S = \{m^2, m^2+1, \dots, (m+1)^2-1\}$. Show that all the products $ab$ with $a, b \in S$ are distinct.

3. Let $N$ be a positive integer for which $N+1$ is a prime and let $a_0, \dots, a_N$ be numbers from the set $\{0, 1\}$, not all equal. A polynomial $f(x)$ with integer coefficients is such that $f(i) = a_i$ for $i = 0, \dots, N$. Prove that the degree of $f$ is at least $N$.

4. For a positive integer $n$, a permutation $(a_0, a_1, \dots, a_n)$ of $0, 1, \dots, n$ is said to be a *square permutation* if $k + a_k$ is a square for each $k = 0, 1, \dots, n$. Show that for every $n$ there exists a square permutation of $0, 1, \dots, n$.

## Fourth Test

Mumbai, May 27

1. In an acute triangle $ABC$ with circumcenter $O$, rays $AO, BO, CO$ meet the sides $BC, CA, AB$ at points $D, E, F$, respectively. Given that the lengths of $OD, OE$ and $OF$ are integers and the circumradius of $\triangle ABC$ is 4, determine the sides $a, b, c$ of $\triangle ABC$.

2. Let $P(x)$ be a polynomial with real coefficients such that $P(x) > 0$ for all $x \geq 0$. Prove that there exists a positive integer $n$ such that $(1+x)^n P(x)$ is a polynomial with nonnegative coefficients.

3. Let $n$ and $p$ be positive integers with $3 \leq p \leq \frac{n}{2}$. Consider an $n$-sided regular polygon with $p$ vertices colored red and the other vertices blue. Show that there are two nondegenerate congruent polygons each with at least $[\frac{p}{2}] + 1$ vertices such that the vertices of one of these polygons are colored red and those of the other coloured blue.

## Fifth Test

Mumbai, May 30

1. Let $P$ be an interior point of a convex quadrilateral $ABCD$. Show that at least one of the four angles $\angle PAB$, $\angle PBC$, $\angle PCD$, $\angle PDA$ is less than or equal to $\pi/4$.



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2. Consider the polynomial $f(z) = (z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)$, where the $\alpha_i$ are complex numbers. Show that there exists a complex number $z_0$ with $|z_0| = 1$ such that
$$|f(z_0)| > \frac{1}{3^n}\prod_{j=1}^{n}(1+|\alpha_j|).$$

3. Prove that for any integers $m \ge n \ge 2$ the number of polynomials of degree $2n-1$ with distinct coefficients from the set $\{1,2,\dots,2m\}$ that are divisible by $x^{n-1} + x^{n-2} + \cdot + x + 1$ is
$$2^n n!\left[\binom{4m+1}{n+1} - \binom{3m}{n}\right].$$



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