14-th Indian Mathematical Olympiad 1999

1. Points D,E,F are taken on the sides BC,CA,AB of an acute-angled triangle ABC such that $AD \perp BC$, $AE = BC$, and CF bisects $\angle C$. Suppose that CF meets AD at M and DE at N such that $FM = 2$, $MN = 1$, and $NC = 3$. Find the perimeter of $\triangle ABC$.

2. In a village 1998 persons volunteered to clean up, for a fair, a rectangular field with integer sides and perimeter equal to 3996 feet. For this purpose, the field was divided into 1998 equal parts. If each part had an integer area, find the length and width of the field.

3. Prove that there are no nonconstant polynomials $p(x)$ and $q(x)$ with integer coefficients such that $p(x)q(x) = x^5 + 2x + 1$.

4. The equilateral triangles ABC and $A_1B_1C_1$ are inscribed in concentric circles Γ and Γ' respectively. If P and P' are arbitrary points on Γ and Γ' respectively, prove that
$$P'A^2 + P'B^2 + P'C^2 = PA'^2 + PB'^2 + PC'^2.$$

5. Show that among any four distinct positive numbers one can choose three, say A,B,C, such that the three quadratic equations
$$\begin{aligned} Bx^2 + x + C &= 0 \\ Cx^2 + x + A &= 0 \\ Ax^2 + x + B &= 0 \end{aligned}$$
either all have real roots or all have non-real roots.

6. For which positive integers n can the set $\{1,2,3,\ldots,4n\}$ be split into n disjoint four-element subsets $\{a,b,c,d\}$ such that in each of them $a = \frac{b+c+d}{3}$?

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