5-th Indian Mathematical Olympiad 1990

1. If the equation $x^4 + px^3 + qx^2 + rx + s = 0$ has four positive real roots, prove that

   (a) $pr - 16r \geq 0$

   (b) $q^2 - 36s \geq 0$

   with equality in each case if and only if the four roots are equal.

2. Find all pairs of nonnegative integers (x,y) satisfying $(xy-7)^2 = x^2 + y^2$.

3. Let $f : \mathbb{N}_0 \to \mathbb{N}_0$ be a function that satisfies

   (i) $x - f(x) = 19[\frac{x}{19}] - 90[\frac{f(x)}{90}]$ for all nonnegative integers x;

   (ii) $1900 < f(1990) < 2000$.

   Find all possible values that $f(1990)$ can take.

4. Determine the number of three-element subsets of $\{1,2,3,\ldots,300\}$ for which the sum of the elements is a multiple of 3.

5. Let a,b,c be the sides of a triangle. Show that the quantity

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}$$

   must lie between $\frac{3}{2}$ and 2. Can equality hold at either limit?

6. In a scalene triangle ABC the angle at A is obtuse. Determine the set of points D lying on the extended line BC for which $AD = \sqrt{BD \cdot CD}$.

7. For any point P lying within a given acute-angled triangle ABC, let D,E,F denote the feet of the perpendiculars from P onto AB,BC,CA respectively. Find the set of all positions of P for which the triangle DEF is isosceles. For which position of P is the triangle DEF equilateral?

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