4-th Indian Mathematical Olympiad 1989

1. Prove that the polynomial
$$f(x) = x^4 + 26x^3 + 52x^2 + 78x + 1989$$
is irreducible over $\mathbb{Z}[x]$.

2. Let a, b, c, d be real numbers, not all zero. Prove that the roots of the polynomial $x^6 + ax^3 + bx^2 + cx + d$ cannot all be real.

3. Let A be a subset of the set $\{1, 11, 21, 31, \dots, 551\}$ whose no two elements add up to 552. Show that A has not more than 28 elements.

4. Find all natural numbers n such that

   (i) n is not a square, and

   (ii) $[\sqrt{n}]^3$ divides n^2.

5. Let a, b, c be the sides of a triangle. Show that the quantity
$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}$$
must lie between $\frac{3}{2}$ and 2. Can the equality hold at either limit?

6. In a scalene triangle ABC the angle at A is obtuse. Determine the set of points on the extended side BC such that $AD = \sqrt{BD \cdot CD}$.

7. A triangle ABC is acute-angled. For any point P within the triangle, D, E, and F denote the projections of P onto BC, CA, AB respectively. Find the locus of P for which triangle DEF is isosceles. When is $\triangle DEF$ equilateral?

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com