

20-th Indian Mathematical Olympiad 2005

1. In a triangle ABC , the median AM intersect the incircle at K and L , where K is between A and L . Prove that if $AK = KL = LM$, then the sides of triangle ABC are in the ratio $5 : 10 : 13$ in some order.
2. If α and β are positive integers such that

$$\frac{43}{197} < \frac{\alpha}{\beta} < \frac{17}{77},$$

find the smallest possible value of β .

3. Suppose that p, q, r are positive numbers, not all equal, such that two of the equations

$$px^2 + 2qx + r = 0, \quad qx^2 + 2rx + p = 0, \quad rx^2 + 2px + q = 0$$

have a common root α . Prove that

- (a) α is real and negative, and
 - (b) the third equation has no real roots.
4. Consider all six-digit numbers whose decimal digits from left to right are in non-increasing order (such as e.g. 877550). If all such numbers are written in increasing order, find the number on the 2005-th position
 5. Given a positive integer x_1 , a sequence $(x_n)_{n=1}^{\infty}$ is constructed as follows: For $n \geq 2$, x_n is obtained from x_{n-1} by adding some nonzero digit of x_{n-1} .
 - (a) Prove that there is an even number in the sequence;
 - (b) Prove that there are infinitely many even numbers in the sequence.
 6. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x^2 + yf(z)) = xf(x) + zf(y) \quad \text{for all } x, y, z \in \mathbb{R}.$$