

## 18-th Indian Mathematical Olympiad 2003

1. Let  $P$  be an interior point of an acute triangle  $ABC$ . The lines  $BP$  and  $CP$  meet  $AC$  and  $AB$  at  $E$  and  $F$  respectively, and the line  $AP$  meets  $EF$  at point  $D$ . Let  $K$  be the projection of  $D$  on  $BC$ . Prove that  $DK$  bisects  $\angle EKF$ .
2. Find all primes  $p, q$  and even numbers  $n > 2$  satisfying the equation

$$p^n + p^{n-1} + \cdots + p + 1 = q^2 + q + 1.$$

3. Show that for every real number  $a$  the equation

$$8x^4 - 16x^3 + 16x^2 - 8x + a = 0$$

has at least one non-real root and find the sum of the non-real roots of this equation.

4. Find all 7-digit numbers formed by using the digits 5 and 7 and divisible by both 5 and 7.
5. Let  $a, b, c$  be the sides of a triangle  $ABC$ . Consider the triangle  $A_1B_1C_1$  with sides  $a + \frac{b}{2}, b + \frac{c}{2}, c + \frac{a}{2}$ . Prove that

$$S_{A_1B_1C_1} \geq \frac{9}{4}S_{ABC},$$

where  $S_X$  denotes the area of triangle  $X$ .

6. In a lottery, tickets are given nine-digit codes using only the digits 1, 2, 3. Each ticket is colored red, blue or green in such a way that two tickets whose codes differ in all the nine places get different colors. Suppose that the tickets with the codes 122222222 and 222222222 are red and green respectively. Find the color of the ticket with the code 123123123.