22-nd Iberoamerican Mathematical Olympiad

September 11–12, 2007

First Day

1. Given an integer m, define the sequence $\{a_n\}$ as follows:

$$a_1 = \frac{m}{2},\ \ a_{n+1} = a_n \cdot \lceil a_n \rceil, \text{ for } n \geq 1.$$

   Find all values of m for which a_{2007} is the first integer appearing in the sequence.

2. Let I be the incenter of $\triangle ABC$ and let Γ be a circle centered at I whose radius is greater than the inradius but does not pass through any of the vertices. Out of two intersection points of Γ and AB, denote by X_1 the one closer to B. Let X_2 and X_3 be the intersections of Γ with BC, with X_2 closer to B; and let X_4 be one of the intersection points of Γ with CA that is closer to C. Let K be the intersection of X_1X_2 and X_3X_4. Prove that AK bisects X_2X_3.

3. Two teams, A and B compete for a territory bounded by a circle. The team A has n blue flags while the team B has n white flags ($n \geq 2$). A starts the game and the teams alternate their moves. In each turn, a team places one of their flags in a point of the circumference of the circle that does not contain a flag already. Once all $2n$ flags are placed, the territory is divided between the teams in the following way: A point x of the territory belongs to team A if all the flags that are closest to x are blue; A point x belongs to B if all the flags that are closest to x are white; If there are both blue and white flags among those that are closet to x, then x doesn't belong to any of the teams. The winner is a team whose union of points cover a larger area. If the areas are the same, the game ends in a draw.

   Prove that the team B has a winning strategy for every n.

Second Day

4. A piece called *dragon* moves in a 19×19 board according to the following rules: It travels by four squares (horizontally or vertically) and then it moves one square more in a direction perpendicular to its previous direction. It is known that by using these moves a dragon can reach each square of he board. The *dragon distance* between two squares is defined as the smallest number of moves needed for a dragon to move from one square to the other.

   Let C be a corner square and V the square that has only one point in common with C. Show that there exists a square X on the board, such that its dragon distance from C is greater than the dragon distance between C and V.

5. A positive integer n is called *nice* if the set of its divisors (including 1 and n) can be split into 3 subsets such that the sum of the elements in each of them is the same. Determine the least number of divisors a *nice* number can have.

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6. Let $\mathscr{F}$ be a family of hexagons H satisfying the following properties:

   (i) H has parallel opposite sides.

   (ii) Any 3 vertices of H can be covered with a strip of width 1.

   Determine the smallest $l \in \mathbb{R}$ such that every hexagon belonging to $\mathscr{F}$ can be covered with a strip of width l.

   Note. A strip is the area bounded by two parallel lines at a distance l. The two lines are also considered to belong to the strip.

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