

50-th International Mathematical Olympiad

Bremen, Germany, July 10–22, 2009

First Day – July 15

1. Let n be a positive integer and let a_1, \dots, a_k ($k \geq 2$) be distinct integers in the set $\{1, \dots, n\}$ such that n divides $a_i(a_{i+1} - 1)$ for $i = 1, \dots, k - 1$. Prove that n does not divide $a_k(a_1 - 1)$.
2. Let ABC be a triangle with circumcenter O . The points P and Q are interior points of the sides CA and AB , respectively. Let K, L and M be the midpoints of the segments BP, CQ , and PQ , respectively, and let Γ be the circle passing through K, L , and M . Suppose that the line PQ is tangent to the circle Γ . Prove that $OP = OQ$.
3. Suppose that s_1, s_2, s_3, \dots is a strictly increasing sequence of positive integers such that the subsequences

$$s_{s_1}, s_{s_2}, s_{s_3}, \dots \quad \text{and} \quad s_{s_1+1}, s_{s_2+1}, s_{s_3+1}, \dots$$

are both arithmetic progressions. Prove that the sequence s_1, s_2, s_3, \dots is itself an arithmetic progression.

Second Day – July 16

4. Let ABC be a triangle with $AB = AC$. The angle bisectors of $\angle CAB$ and $\angle ABC$ meet the sides BC and CA at D and E , respectively. Let K be the incenter of triangle ADC . Suppose that $\angle BEK = 45^\circ$. Find all possible values of $\angle CAB$.
5. Determine all functions f from the set of positive integers to the set of positive integers such that, for all positive integers a and b , there exists a non-degenerate triangle with sides of lengths

$$a, f(b) \quad \text{and} \quad f(b + f(a) - 1).$$

(A triangle is *non-degenerate* if its vertices are not collinear.)

6. Let a_1, a_2, \dots, a_n be distinct positive integers and let M be a set of $n - 1$ positive integers not containing $s = a_1 + a_2 + \dots + a_n$. A grasshopper is to jump along the real axis, starting at the point 0 and making n jumps to the right with lengths a_1, a_2, \dots, a_n in some order. Prove that the order can be chosen in such a way that the grasshopper never lands on any point in M .