

# 50-th International Mathematical Olympiad

Bremen, Germany, July 10–22, 2009

*First Day – July 15*

1. Let  $n$  be a positive integer and let  $a_1, \dots, a_k$  ( $k \geq 2$ ) be distinct integers in the set  $\{1, \dots, n\}$  such that  $n$  divides  $a_i(a_{i+1} - 1)$  for  $i = 1, \dots, k - 1$ . Prove that  $n$  does not divide  $a_k(a_1 - 1)$ .
2. Let  $ABC$  be a triangle with circumcenter  $O$ . The points  $P$  and  $Q$  are interior points of the sides  $CA$  and  $AB$ , respectively. Let  $K, L$  and  $M$  be the midpoints of the segments  $BP, CQ$ , and  $PQ$ , respectively, and let  $\Gamma$  be the circle passing through  $K, L$ , and  $M$ . Suppose that the line  $PQ$  is tangent to the circle  $\Gamma$ . Prove that  $OP = OQ$ .
3. Suppose that  $s_1, s_2, s_3, \dots$  is a strictly increasing sequence of positive integers such that the subsequences

$$s_{s_1}, s_{s_2}, s_{s_3}, \dots \quad \text{and} \quad s_{s_1+1}, s_{s_2+1}, s_{s_3+1}, \dots$$

are both arithmetic progressions. Prove that the sequence  $s_1, s_2, s_3, \dots$  is itself an arithmetic progression.

*Second Day – July 16*

4. Let  $ABC$  be a triangle with  $AB = AC$ . The angle bisectors of  $\angle CAB$  and  $\angle ABC$  meet the sides  $BC$  and  $CA$  at  $D$  and  $E$ , respectively. Let  $K$  be the incenter of triangle  $ADC$ . Suppose that  $\angle BEK = 45^\circ$ . Find all possible values of  $\angle CAB$ .
5. Determine all functions  $f$  from the set of positive integers to the set of positive integers such that, for all positive integers  $a$  and  $b$ , there exists a non-degenerate triangle with sides of lengths

$$a, f(b) \quad \text{and} \quad f(b + f(a) - 1).$$

(A triangle is *non-degenerate* if its vertices are not collinear.)

6. Let  $a_1, a_2, \dots, a_n$  be distinct positive integers and let  $M$  be a set of  $n - 1$  positive integers not containing  $s = a_1 + a_2 + \dots + a_n$ . A grasshopper is to jump along the real axis, starting at the point 0 and making  $n$  jumps to the right with lengths  $a_1, a_2, \dots, a_n$  in some order. Prove that the order can be chosen in such a way that the grasshopper never lands on any point in  $M$ .