49-th International Mathematical Olympiad

Madrid, Spain, July 10–22, 2008

First Day – July 16

1. An acute-angled triangle ABC has orthocenter H. The circle passing through H with center at the midpoint of BC intersects the line BC at A_1 and A_2. Similarly, the circle passing through H with center at the midpoint of CA intersects the line CA at B_1 and B_2, and the circle passing through H with center at the midpoint of AB intersects the line AB at C_1 and C_2. Show that A_1, A_2, B_1, B_2, C_1, C_2 lie on a circle. *(Russia)*

2. (a) Prove that
$$\frac{x^2}{(x-1)^2}+\frac{y^2}{(y-1)^2}+\frac{z^2}{(z-1)^2}\geq 1$$
for all real number x, y, z, each different from 1, and satisfying $xyz=1$.

   (b) Prove that equality holds above for infinitely many triples of rational numbers x, y, z, each different from 1, satisfying $xyz=1$.

   (Austria)

3. Prove that there exist infinitely many positive integers n such that n^2+1 has a prime divisor which is greater than $2n+\sqrt{2n}$. *(Lithuania)*

Second Day – July 17

4. Find all functions $f:(0,+\infty)\to(0,+\infty)$ (so, f is a function from the positive real numbers to the positive real numbers) such that
$$\frac{(f(w))^2+(f(x))^2}{f(y^2)+f(z^2)}=\frac{w^2+x^2}{y^2+z^2}$$
for all positive real numbers w, x, y, z, satisfying $wx=yz$. *(South Korea)*

5. Let n and k be positive integers with $k\geq n$ and $k-n$ an even number. Let $2n$ lamps labelled 1, 2, ..., $2n$ be given, each of which can be either *on* or *off*. Initially all the lamps are off. We consider sequence of *steps*: at each step one of the lamps is switched (from on to off or from off to on).

   Let N be the number of such sequences consisting of k steps and resulting in the state where lamps 1 through n are all on, and lamps $n+1$ through $2n$ are all off, but where none of the lampos $n+1$ through $2n$ is ever switched on.

   Determine the ration N/M. *(France)*

6. Let $ABCD$ be a convex quadrilateral with $|BA|\neq|BC|$. Denote the incircles of triangles ABC and ADC by ω_1 and ω_2 respectively. Suppose that there exists a circle ω tangent to the ray BA beyond A and to the ray BC beyond C, which is also tangent to the lines AD and CD. Prove that the common external tangents of ω_1 and ω_2 intersect on ω. *(Russia)*

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