45-th International Mathematical Olympiad

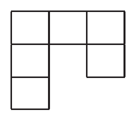
Athens, Greece, July 7–19, 2004

First Day – July 12

1. Let ABC be an acute-angled triangle with $AB \neq AC$. The circle with diameter BC intersects the sides AB and AC at M and N, respectively. Denote by O the midpoint of BC. The bisectors of the angles BAC and MON intersect at R. Prove that the circumcircles of the triangles BMR and CNR have a common point lying on the line segment BC.
(Romania)

2. Find all polynomials $P(x)$ with real coefficients that satisfy the equality
$$P(a-b)+P(b-c)+P(c-a)=2P(a+b+c)$$
for all triples a,b,c of real numbers such that $ab+bc+ca=0$.
(South Korea)

3. Determine all $m \times n$ rectangles that can be covered with *hooks* made up of 6 unit squares, as in the figure:



Rotations and reflections of hooks are allowed. The rectangle must be covered without gaps and overlaps. No part of a hook may cover area outside the rectangle.
(Estonia)

Second Day – July 13

4. Let $n \geq 3$ be an integer and $t_1, t_2, \ldots, t_n$ positive real numbers such that
$$n^2+1 > (t_1+t_2+\cdots+t_n)\left(\frac{1}{t_1}+\frac{1}{t_2}+\cdots+\frac{1}{t_n}\right).$$
Show that t_i, t_j, t_k are the side lengths of a triangle for all i, j, k with $1 \leq i < j < k \leq n$.
(South Korea)

5. In a convex quadrilateral $ABCD$ the diagonal BD does not bisect the angles ABC and CDA. The point P lies inside $ABCD$ and satisfies
$$\angle PBC = \angle DBA \quad \text{and} \quad \angle PDC = \angle BDA.$$
Prove that $ABCD$ is a cyclic quadrilateral if and only if $AP = CP$.
(Poland)

6. We call a positive integer *alternate* if its decimal digits are alternately odd and even. Find all positive integers n such that n has an alternate multiple.
(Iran)

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