44-th International Mathematical Olympiad

Tokyo, Japan, July 7–19, 2003

First Day – July 13

1. Let A be a 101-element subset of the set $S = \{1, 2, \dots, 1000000\}$. Prove that there exist numbers $t_1, t_2, \dots, t_{100}$ in S such that the sets

$$A_j = \{x + t_j | x \in A\}, \qquad j = 1, 2, \dots, 100,$$

are pairwise disjoint. *(Brazil)*

2. Determine all pairs (a, b) of positive integers such that

$$\frac{a^2}{2ab^2 - b^3 + 1}$$

is a positive integer. *(Bulgaria)*

3. Each pair of opposite sides of a convex hexagon has the following property: The distance between their midpoints is equal to $\sqrt{3}/2$ times the sum of their lengths. Prove that all the angles of the hexagon are equal. *(Poland)*

Second Day – July 14

4. Let $ABCD$ be a cyclic quadrilateral. Let P, Q, R be the feet of the perpendiculars from D to the lines BC, CA, AB, respectively. Show that $PQ = QR$ if and only if the bisectors of $\angle ABC$ and $\angle ADC$ are concurrent with AC. *(Finland)*

5. Let n be a positive integer and let $x_1 \le x_2 \le \cdots \le x_n$ be real numbers.

   (a) Prove that

$$\left(\sum_{i,j=1}^{n} |x_i - x_j| \right)^2 \le \frac{2(n^2 - 1)}{3} \sum_{i,j=1}^{n} (x_i - x_j)^2.$$

   (b) Show that equality holds if and only if $x_1, \dots, x_n$ is an arithmetic progression. *(Ireland)*

6. Let p be a prime number. Prove that there exists a prime number q such that for every integer n, the number $n^p - p$ is not divisible by q. *(France)*

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