34-th International Mathematical Olympiad

Istanbul, Turkey, July 13–24, 1993

First Day – July 18

1. Let $n > 1$ be an integer and let $f(x) = x^n + 5x^{n-1} + 3$. Prove that there do not exist polynomials $g(x), h(x)$, each having integer coefficients and degree at least one, such that $f(x) = g(x)h(x)$. *(Ireland)*

2. A,B,C,D are four points in the plane, with C,D on the same side of the line AB, such that $AC \cdot BD = AD \cdot BC$ and $\measuredangle ADB = 90^\circ + \measuredangle ACB$. Find the ratio $\dfrac{AB \cdot CD}{AC \cdot BD}$, and prove that circles ACD, BCD are orthogonal. *(Great Britain)*

3. On an infinite chessboard, a solitaire game is played as follows: At the start, we have n^2 pieces occupying n^2 squares that form a square of side n. The only allowed move is a jump horizontally or vertically over an occupied square to an unoccupied one, and the piece that has been jumped over is removed. For what positive integers n can the game end with only one piece remaining on the board? *(Finland)*

Second Day – July 19

4. For three points A,B,C in the plane we define $m(ABC)$ to be the smallest length of the three altitudes of the triangle ABC, where in the case of A,B,C collinear, $m(ABC) = 0$. Let A,B,C be given points in the plane. Prove that for any point X in the plane,
$$m(ABC) \le m(ABX) + m(AXC) + m(XBC).$$
(FYR Macedonia)

5. Let $\mathbb{N} = \{1,2,3,\dots\}$. Determine whether there exists a strictly increasing function $f : \mathbb{N} \to \mathbb{N}$ with the following properties:
$$f(1) = 2 \quad \text{and} \quad f(f(n)) = f(n) + n \ (n \in \mathbb{N}).$$
(Germany)

6. Let n be an integer greater than 1. In a circular arrangement of n lamps $L_0, \dots, L_{n-1}$, each one of that can be either ON or OFF, we start with the situation where all lamps are ON, and then carry out a sequence of steps, $Step_0, Step_1, \dots$. If L_{j-1} (j is taken $\bmod n$) is ON, then $Step_j$ changes the status of L_j (it goes from ON to OFF or from OFF to ON) but does not change the status of any of the other lamps. If L_{j-1} is OFF, then $Step_j$ does not change anything at all. Show that:

   (a) There is a positive integer $M(n)$ such that after $M(n)$ steps all lamps are ON again.

   (b) If n has the form 2^k, then all lamps are ON after $n^2 - 1$ steps.

   (c) If n has the form $2^k + 1$, then all lamps are ON after $n^2 - n + 1$ steps. *(Netherlands)*

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