31-st International Mathematical Olympiad

Beijing, China, July 8–19, 1990

First Day – July 12

1. Given a circle with two chords AB, CD that meet at E, let M be a point of chord AB other than E. Draw the circle through D, E, and M. The tangent line to the circle DEM at E meets the lines BC, AC at F, G, respectively. Given $\frac{AM}{AB} = \lambda$, find $\frac{GE}{EF}$. *(India)*

2. On a circle, $2n-1$ $(n \geq 3)$ different points are given. Find the minimal natural number N with the property that whenever N of the given points are colored black, there exist two black points such that the interior of one of the corresponding arcs contains exactly n of the given $2n-1$ points. *(Czechoslovakia)*

3. Find all positive integers n having the property that $\frac{2^n+1}{n^2}$ is an integer. *(Romania)*

Second Day – July 13

4. Let $\mathbb{Q}^+$ be the set of positive rational numbers. Construct a function $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that
$$f(xf(y)) = \frac{f(x)}{y}, \qquad \text{for all } x, y \text{ in } \mathbb{Q}^+.$$
(Turkey)

5. Two players A and B play a game in which they choose numbers alternately according to the following rule: At the beginning, an initial natural number $n_0 > 1$ is given. Knowing n_{2k}, player A may choose any $n_{2k+1} \in \mathbb{N}$ such that $n_{2k} \leq n_{2k+1} \leq n_{2k}^2$. Then player B chooses a number $n_{2k+2} \in \mathbb{N}$ such that $\frac{n_{2k+1}}{n_{2k+2}} = p^r$, where p is a prime number and $r \in \mathbb{N}$.

   It is stipulated that player A wins the game if he (she) succeeds in choosing the number 1990, and player B wins if he (she) succeeds in choosing 1. For which natural numbers n_0 can player A manage to win the game, for which n_0 can player B manage to win, and for which n_0 can players A and B each force a tie? *(FR Germany)*

6. Is there a 1990-gon with the following properties (i) and (ii)?

   (i) All angles are equal;

   (ii) The lengths of the 1990 sides are a permutation of the numbers $1^2, 2^2, \ldots, 1989^2, 1990^2$. *(Netherlands)*

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