29-th International Mathematical Olympiad

Canberra, Australia, July 9–21, 1988

First Day – July 15

1. Consider two concentric circles of radii R and r $(R > r)$ with center O. Fix P on the small circle and consider the variable chord PA of the small circle. Points B and C lie on the large circle; B, P, C are collinear and BC is perpendicular to AP.

   (a) For which value(s) of $\angle OPA$ is the sum $BC^2 + CA^2 + AB^2$ extremal?

   (b) What are the possible positions of the midpoints U of BA and V of AC as $\measuredangle OPA$ varies? *(Luxembourg)*

2. Let n be an even positive integer. Let $A_1, A_2, \ldots, A_{n+1}$ be sets having n elements each such that any two of them have exactly one element in common, while every element of their union belongs to at least two of the given sets. For which n can one assign to every element of the union one of the numbers 0 and 1 in such a manner that each of the sets has exactly $n/2$ zeros? *(Czechoslovakia)*

3. A function f defined on the positive integers (and taking positive integer values) is given by
$$f(1) = 1, \quad f(3) = 3,$$
$$f(2n) = f(n),$$
$$f(4n+1) = 2f(2n+1) - f(n),$$
$$f(4n+3) = 3f(2n+1) - 2f(n),$$
for all positive integers n. Determine with proof the number of positive integers less than or equal to 1988 for which $f(n) = n$. *(Great Britain)*

Second Day – July 16

4. Show that the solution set of the inequality
$$\sum_{k=1}^{70} \frac{k}{x-k} \geq \frac{5}{4}$$
is the union of disjoint half-open intervals with the sum of lengths 1988. *(Ireland)*

5. In a right-angled triangle ABC let AD be the altitude drawn to the hypotenuse and let the straight line joining the incenters of the triangles ABD, ACD intersect the sides AB, AC at the points K, L respectively. If E and E_1 denote the areas of the triangles ABC and AKL respectively, show that $\dfrac{E}{E_1} \geq 2$. *(Greece)*

6. Let a and b be two positive integers such that $ab+1$ divides a^2+b^2. Show that $\dfrac{a^2+b^2}{ab+1}$ is a perfect square. *(FR Germany)*

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