28-th International Mathematical Olympiad

Havana, Cuba, July 5–16, 1987

First Day – July 10

1. Let S be a set of n elements. We denote the number of all permutations of S that have exactly k fixed points by $p_n(k)$. Prove that
$$\sum_{k=0}^{n} kp_n(k) = n!.$$
(FR Germany)

2. The prolongation of the bisector AL ($L \in BC$) in the acute-angled triangle ABC intersects the circumscribed circle at point N. From point L to the sides AB and AC are drawn the perpendiculars LK and LM respectively. Prove that the area of the triangle ABC is equal to the area of the quadrilateral $AKNM$. *(Soviet Union)*

3. Suppose $x_1, x_2, \dots, x_n$ are real numbers with $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$. Prove that for any integer $k > 1$ there are integers e_i not all 0 and with $|e_i| < k$ such that
$$|e_1x_1 + e_2x_2 + \cdots + e_nx_n| \le \frac{(k-1)\sqrt{n}}{k^n - 1}.$$
(Germany)

Second Day – July 11

4. Does there exist a function $f : \mathbb{N} \to \mathbb{N}$, such that $f(f(n)) = n + 1987$ for every natural number n? *(Vietnam)*

5. Prove that for every natural number $n \ge 3$ it is possible to put n points in the Euclidean plane such that the distance between each pair of points is irrational and each three points determine a nondegenerate triangle with rational area. *(DR Germany)*

6. Let $f(x) = x^2 + x + p$, $p \in \mathbb{N}$. Prove that if the numbers $f(0), f(1), \dots, f([\sqrt{p/3}\,])$ are primes, then all the numbers $f(0), f(1), \dots, f(p-2)$ are primes. *(Soviet Union)*

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