27-th International Mathematical Olympiad

Warsaw, Poland, July 4–15, 1986

First Day – July 9

1. The set $S=\{2,5,13\}$ has the property that for every $a,b\in S$, $a\neq b$, the number $ab-1$ is a perfect square. Show that for every positive integer d not in S, the set $S\cup\{d\}$ does not have the above property.
(FR Germany)

2. Let A,B,C be fixed points in the plane. A man starts from a certain point P_0 and walks directly to A. At A he turns his direction by 60° to the left and walks to P_1 such that $P_0A=AP_1$. After he performs the same action 1986 times successively around the points $A,B,C,A,B,C,\dots$, he returns to the starting point. Prove that ABC is an equilateral triangle, and that the vertices A,B,C are arranged counterclockwise. *(China)*

3. To each vertex P_i $(i=1,\dots,5)$ of a pentagon an integer x_i is assigned, the sum $s=\sum x_i$ being positive. The following operation is allowed, provided at least one of the x_i's is negative: Choose a negative x_i, replace it by $-x_i$, and add the former value of x_i to the integers assigned to the two neighboring vertices of P_i (the remaining two integers are left unchanged).

   This operation is to be performed repeatedly until all negative integers disappear. Decide whether this procedure must eventually terminate.
(DR Germany)

Second Day – July 10

4. Let A,B be adjacent vertices of a regular n-gon in the plane and let O be its center. Now let the triangle ABO glide around the polygon in such a way that the points A and B move along the whole circumference of the polygon. Describe the figure traced by the vertex O. *(Israel)*

5. Find, with proof, all functions f defined on the nonnegative real numbers and taking nonnegative real values such that

   (i) $f[xf(y)]f(y)=f(x+y)$,

   (ii) $f(2)=0$ but $f(x)\neq 0$ for $0\le x<2$. *(Great Britain)*

6. Prove or disprove: Given a finite set of points with integer coefficients in the plane, it is possible to color some of these points red and the remaining ones white in such a way that for any straight line L parallel to one of the coordinate axes, the number of red colored points and the number of white colored points on L differ by at most 1. *(DR Germany)*

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