25-th International Mathematical Olympiad

Prague, Czechoslovakia, June 29–July 10, 1984

First Day – July 4

1. Let x,y,z be nonnegative real numbers with $x+y+z=1$. Show that
$$0 \leq xy+yz+zx-2xyz \leq \frac{7}{27}.$$
(FR Germany)

2. Find two positive integers a,b such that none of the numbers $a,b,a+b$ is divisible by 7 and $(a+b)^7-a^7-b^7$ is divisible by 7^7. *(Netherlands)*

3. In a plane two different points O and A are given. For each point $X \neq O$ of the plane denote by $\alpha(X)$ the angle AOX measured in radians $(0 \leq \alpha(X) < 2\pi)$ and by $C(X)$ the circle with center O and radius $OX + \frac{\alpha(X)}{OX}$. Suppose each point of the plane is colored by one of a finite number of colors. Show that there exists a point X with $\alpha(X) > 0$ such that its color appears somewhere on the circle $C(X)$.
(Romania)

Second Day – July 5

4. Let $ABCD$ be a convex quadrilateral for which the circle of diameter AB is tangent to the line CD. Show that the circle of diameter CD is tangent to the line AB if and only if the lines BC and AD are parallel.
(Romania)

5. Let d be the sum of the lengths of all diagonals of a convex polygon of n $(n>3)$ vertices, and let p be its perimeter. Prove that
$$\frac{n-3}{2} < \frac{d}{p} < \frac{1}{2}\left(\left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right]-2\right).$$
(Mongolia)

6. Let a,b,c,d be odd positive integers such that $a<b<c<d$, $ad=bc$, and $a+d=2^k$, $b+c=2^m$ for some integers k and m. Prove that $a=1$.
(Poland)

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