24-th International Mathematical Olympiad

Paris, France, July 1–12, 1983

First Day – July 6

1. Find all functions f defined on the positive real numbers and taking positive real values that satisfy the following conditions:

   (i) $f(xf(y)) = yf(x)$ for all positive real x, y;

   (ii) $f(x) \to 0$ as $x \to +\infty$. *(Great Britain)*

2. Let K be one of the two intersection points of the circles W_1 and W_2. Let O_1 and O_2 be the centers of W_1 and W_2. The two common tangents to the circles meet W_1 and W_2 respectively in P_1 and P_2, the first tangent, and Q_1 and Q_2 the second tangent. Let M_1 and M_2 be the midpoints of P_1Q_1 and P_2Q_2, respectively. Prove that $\angle O_1KO_2 = \angle M_1KM_2$.

   (Soviet Union)

3. Let a, b, c be positive integers satisfying $(a,b) = (b,c) = (c,a) = 1$. Show that $2abc - ab - bc - ca$ is the largest integer not representable as

   $$xbc + yca + zab$$

   with nonnegative integers x, y, z. *(FR Germany)*

Second Day – July 7

4. Let ABC be an equilateral triangle. Let E be the set of all points from segments AB, BC, and CA (including A, B, and C). Is it true that for any partition of the set E into two disjoint subsets, there exists a right-angled triangle all of whose vertices belong to the same subset in the partition?

   (Belgium)

5. Prove or disprove the following statement: In the set $\{1, 2, 3, \dots, 10^5\}$ a subset of 1983 elements can be found that does not contain any three consecutive terms of an arithmetic progression. *(Poland)*

6. If a, b, and c are sides of a triangle, prove that

   $$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geq 0$$

   and determine when there is equality. *(United States of America)*

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com