23-rd International Mathematical Olympiad

Budapest, Hungary, July 5–14, 1982

First Day – July 9

1. The function $f(n)$ is defined for all positive integers n and takes on nonnegative integer values. Also, for all m,n,

$$f(m+n)-f(m)-f(n)=0 \quad \text{or} \quad 1;$$

$$f(2)=0, \quad f(3)>0, \quad \text{and} \quad f(9999)=3333.$$

Determine $f(1982)$. *(Great Britain)*

2. A nonisosceles triangle $A_1A_2A_3$ is given with sides a_1,a_2,a_3 (a_i is the side opposite to A_i). For all $i=1,2,3$, M_i is the midpoint of side a_i, T_i is the point where the incircle touches side a_i, and the reflection of T_i in the interior bisector of A_i yields the point S_i. Prove that the lines M_1S_1, M_2S_2, and M_3S_3 are concurrent. *(Netherlands)*

3. Consider the infinite sequences $\{x_n\}$ of positive real numbers such that $x_0=1$ and for all $i\geq 0$, $x_{i+1}\leq x_i$.

   (a) Prove that for every such sequence there is an $n\geq 1$ such that

$$\frac{x_0^2}{x_1}+\frac{x_1^2}{x_2}+\cdots+\frac{x_{n-1}^2}{x_n}\geq 3.999.$$

   (b) Find such a sequence for which $\frac{x_0^2}{x_1}+\frac{x_1^2}{x_2}+\cdots+\frac{x_{n-1}^2}{x_n}<4$ for all n.

(Soviet Union)

Second Day – July 10

4. Prove that if n is a positive integer such that the equation $x^3-3xy^2+y^3=n$ has a solution in integers (x,y), then it has at least three such solutions. Show that the equation has no solution in integers when $n=2891$.

(Great Britain)

5. The diagonals AC and CE of the regular hexagon $ABCDEF$ are divided by the inner points M and N, respectively, so that $\frac{AM}{AC}=\frac{CN}{CE}=r$. Determine r if B, M, and N are collinear. *(Netherlands)*

6. Let S be a square with sides of length 100 and let L be a path within S that does not meet itself and that is composed of linear segments $A_0A_1, A_1A_2, \dots, A_{n-1}A_n$ with $A_0\neq A_n$. Suppose that for every point P of the boundary of S there is a point of L at a distance from P not greater than $\frac{1}{2}$. Prove that there are two points X and Y in L such that the distance between X and Y is not greater than 1 and the length of the part of L that lies between X and Y is not smaller than 198. *(Vietnam)*

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