19-th International Mathematical Olympiad

Belgrade – Arandjelovac, Yugoslavia, July 1–13, 1977

First Day – July 6

1. Equilateral triangles ABK, BCL, CDM, DAN are constructed inside the square $ABCD$. Prove that the midpoints of the four segments KL, LM, MN, NK and the midpoints of the eight segments AK, BK, BL, CL, CM, DM, DN, AN are the twelve vertices of a regular dodecagon.

   (Netherlands)

2. In a finite sequence of real numbers the sum of any seven successive terms is negative, and the sum of any eleven successive terms is positive. Determine the maximum number of terms in the sequence. *(Vietnam)*

3. Let n be a given integer greater than 2, and let V_n be the set of integers $1+kn$, where $k=1,2,\dots$. A number $m \in V_n$ is called indecomposable in V_n if there do not exist numbers $p,q \in V_n$ such that $pq=m$. Prove that there exists a number $r \in V_n$ that can be expressed as the product of elements indecomposable in V_n in more than one way. (Expressions that differ only in order of the elements of V_n will be considered the same.)

   (Netherlands)

Second Day – July 7

4. Let a,b,A,B be given constant real numbers and
$$f(x) = 1 - a\cos x - b\sin x - A\cos 2x - B\sin 2x.$$
   Prove that if $f(x) \geq 0$ for all real x, then
$$a^2+b^2 \leq 2 \quad \text{and} \quad A^2+B^2 \leq 1.$$
   (Great Britain)

5. Let a and b be natural numbers and let q and r be the quotient and remainder respectively when a^2+b^2 is divided by $a+b$. Determine the numbers a and b if $q^2+r=1977$. *(DR Germany)*

6. Let $f:\mathbb{N}\to\mathbb{N}$ be a function that satisfies the inequality $f(n+1) > f(f(n))$ for all $n \in \mathbb{N}$. Prove that $f(n)=n$ for all natural numbers n. *(Bulgaria)*

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