18-th International Mathematical Olympiad

Wienna – Linz, Austria, 1976

First Day – July 12

1. In a convex quadrangle with area $32\,\text{cm}^2$, the sum of the lengths of two nonadjacent edges and of the length of one diagonal is equal to $16\,\text{cm}$. What is the length of the other diagonal? *(Czechoslovakia)*

2. Let $P_1(x) = x^2 - 2$, $P_j(x) = P_1(P_{j-1}(x))$, $j = 2, 3, \dots$. Show that for arbitrary n, the roots of the equation $P_n(x) = x$ are real and different. *(Finland)*

3. A rectangular box can be filled completely with unit cubes. If one places cubes with volume 2 in the box such that their edges are parallel to the edges of the box, one can fill exactly 40% of the box. Determine all possible (interior) sizes of the box. *(Netherlands)*

Second Day – July 13

4. Find the largest number obtainable as the product of positive integers whose sum is 1976. *(United States of America)*

5. Let a set of p equations be given,
$$\begin{aligned} a_{11}x_1 + \dots + a_{1q}x_q &= 0, \\ a_{21}x_1 + \dots + a_{2q}x_q &= 0, \\ &\vdots \\ a_{p1}x_1 + \dots + a_{pq}x_q &= 0, \end{aligned}$$
with coefficients a_{ij} satisfying $a_{ij} = -1$, 0, or $+1$ for all $i = 1, \dots, p$ and $j = 1, \dots, q$. Prove that if $q = 2p$, there exists a solution $x_1, \dots, x_q$ of this system such that all x_j $(j = 1, \dots, q)$ are integers satisfying $|x_j| \le q$ and $x_j \ne 0$ for at least one value of j. *(Netherlands)*

6. For all positive integral n, $u_{n+1} = u_n(u_{n-1}^2 - 2) - u_1$, $u_0 = 2$, and $u_1 = 2\frac{1}{2}$. Prove that
$$3 \log_2 [u_n] = 2^n - (-1)^n,$$
where $[x]$ is the integral part of x. *(Great Britain)*

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