17-th International Mathematical Olympiad

Burgas – Sofia, Bulgaria, 1975

First Day – July 7

1. Let $x_1 \geq x_2 \geq \cdots \geq x_n$ and $y_1 \geq y_2 \geq \cdots \geq y_n$ be two n-tuples of numbers. Prove that
$$\sum_{i=1}^{n}(x_i - y_i)^2 \leq \sum_{i=1}^{n}(x_i - z_i)^2$$
is true when $z_1, z_2, \ldots, z_n$ denote $y_1, y_2, \ldots, y_n$ taken in another order.
(Czechoslovakia)

2. Let $a_1, a_2, a_3, \ldots$ be any infinite increasing sequence of positive integers. (For every integer $i > 0$, $a_{i+1} > a_i$.) Prove that there are infinitely many m for which positive integers x, y, h, k can be found such that $0 < h < k < m$ and $a_m = xa_h + ya_k$.
(Great Britain)

3. On the sides of an arbitrary triangle ABC, triangles BPC, CQA, and ARB are externally erected such that
$$\measuredangle PBC = \measuredangle CAQ = 45^\circ,$$
$$\measuredangle BCP = \measuredangle QCA = 30^\circ,$$
$$\measuredangle ABR = \measuredangle BAR = 15^\circ.$$
Prove that $\measuredangle QRP = 90^\circ$ and $QR = RP$.
(Netherlands)

Second Day – July 8

4. Let A be the sum of the digits of the number 4444^{4444} and B the sum of the digits of the number A. Find the sum of the digits of the number B.
(Soviet Union)

5. Is it possible to plot 1975 points on a circle with radius 1 so that the distance between any two of them is a rational number (distances have to be measured by chords)?
(Soviet Union)

6. The function $f(x, y)$ is a homogeneous polynomial of the nth degree in x and y. If $f(1, 0) = 1$ and for all a, b, c,
$$f(a + b, c) + f(b + c, a) + f(c + a, b) = 0,$$
prove that $f(x, y) = (x - 2y)(x + y)^{n-1}$.
(Great Britain)

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