15-th International Mathematical Olympiad

Moscow, Soviet Union, July 5–16, 1973

First Day – July 9

1. Let O be a point on the line l and $\overrightarrow{OP_1}, \overrightarrow{OP_2}, \dots, \overrightarrow{OP_n}$ unit vectors such that points $P_1, P_2, \dots, P_n$ and line l lie in the same plane and all points P_i lie in the same half-plane determined by l. Prove that if n is odd, then
$$\left|\overrightarrow{OP_1} + \overrightarrow{OP_2} + \cdots + \overrightarrow{OP_n}\right| \geq 1.$$
(Czechoslovakia)

2. Does there exist a finite set M of points in space, not all in the same plane, such that for each two points $A, B \in M$ there exist two other points $C, D \in M$ such that lines AB and CD are parallel but not equal?
(Poland)

3. Determine the minimum of $a^2 + b^2$ if a and b are real numbers for which the equation
$$x^4 + ax^3 + bx^2 + ax + 1 = 0$$
has at least one real solution. *(Sweden)*

Second Day – July 10

4. A soldier has to investigate whether there are mines in an area that has the form of equilateral triangle. The radius of his detector's range is equal to one-half the altitude of the triangle. The soldier starts from one vertex of the triangle. Determine the smallest path through which the soldier has to pass in order to check the entire region. *(Yugoslavia)*

5. Let G be the set of functions $f : \mathbb{R} \to \mathbb{R}$ of the form $f(x) = ax + b$, where a and b are real numbers and $a \neq 0$. Suppose that G satisfies the following conditions:
   (1) If $f, g \in G$, then $g \circ f \in G$, where $(g \circ f)(x) = g[f(x)]$.
   (2) If $f \in G$ then the inverse f^{-1} of f exists and belongs to G.
   (3) For each $f \in G$ there exists a number $x_f \in \mathbb{R}$ such that $f(x_f) = x_f$.

   Prove that there exists a number $k \in \mathbb{R}$ such that $f(k) = k$ for all $f \in G$.
(Poland)

6. Let $a_1, a_2, \dots, a_n$ be positive numbers and q a given real number, $0 < q < 1$. Find n real numbers $b_1, b_2, \dots, b_n$ that satisfy:
   (1) $a_k < b_k$ for all $k = 1, 2, \dots, n$;
   (2) $q < \dfrac{b_{k+1}}{b_k} < \dfrac{1}{q}$ for all $k = 1, 2, \dots, n-1$;
   (3) $b_1 + b_2 + \cdots + b_n < \dfrac{1+q}{1-q}(a_1 + a_2 + \cdots + a_n)$. *(Sweden)*

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