9-th International Mathematical Olympiad

Cetinje, Yugoslavia, July 2–13, 1967

First Day – July 5

1. $ABCD$ is a parallelogram; $AB = a$, $AD = 1$, α is the size of $\angle DAB$, and the three angles of the triangle ABD are acute. Prove that the four circles K_A, K_B, K_C, K_D, each of radius 1, whose centers are the vertices A, B, C, D, cover the parallelogram if and only if $a \le \cos\alpha + \sqrt{3}\sin\alpha$.
(Czechoslovakia)

2. Exactly one side of a tetrahedron is of length greater than 1. Show that its volume is less than or equal to $1/8$. *(Poland)*

3. Let k, m, and n be positive integers such that $m+k+1$ is a prime number greater than $n+1$. Write c_s for $s(s+1)$. Prove that the product $(c_{m+1}-c_k)(c_{m+2}-c_k)\cdots(c_{m+n}-c_k)$ is divisible by the product $c_1c_2\cdots c_n$. *(Great Britain)*

Second Day – July 6

4. The triangles $A_0B_0C_0$ and $A'B'C'$ have all their angles acute. Describe how to construct one of the triangles ABC, similar to $A'B'C'$ and circumscribing $A_0B_0C_0$ (so that A, B, C correspond to A', B', C', and AB passes through C_0, BC through A_0, and CA through B_0). Among these triangles ABC describe, and prove, how to construct the triangle with the maximum area. *(Italy)*

5. Consider the sequence (c_n):

$$\begin{array}{ccc} c_1 & = & a_1+a_2+\cdots+a_8, \\ c_2 & = & a_1^2+a_2^2+\cdots+a_8^2, \\ \ldots & & \ldots\ldots\ldots \\ c_n & = & a_1^n+a_2^n+\cdots+a_8^n, \\ \ldots & & \ldots\ldots\ldots \end{array}$$

where $a_1, a_2, \ldots, a_8$ are real numbers, not all equal to zero. Given that among the numbers of the sequence (c_n) there are infinitely many equal to zero, determine all the values of n for which $c_n = 0$. *(Soviet Union)*

6. In a sports competition lasting n days there are m medals to be won. On the first day, one medal and $1/7$ of the remaining $m-1$ medals are won. On the second day, 2 medals and $1/7$ of the remainder are won. And so on. On the nth day exactly n medals are won. How many days did the competition last and what was the total number of medals? *(Hungary)*

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