7-th International Mathematical Olympiad

Berlin, DR Germany, July 3–13, 1965

First Day

1. Find all real numbers $x \in [0, 2\pi]$ such that
$$2\cos x \le |\sqrt{1+\sin 2x} - \sqrt{1-\sin 2x}| \le \sqrt{2}.$$
(Yugoslavia)

2. Consider the system of equations
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= 0, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= 0, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= 0, \end{cases}$$
whose coefficients satisfy the following conditions:
   (a) a_{11}, a_{22}, a_{33} are positive real numbers;
   (b) all other coefficients are negative;
   (c) in each of the equations the sum of the coefficients is positive.

   Prove that $x_1 = x_2 = x_3 = 0$ is the only solution to the system. *(Poland)*

3. A tetrahedron $ABCD$ is given. The lengths of the edges AB and CD are a and b, respectively, the distance between the lines AB and CD is d, and the angle between them is equal to ω. The tetrahedron is divided into two parts by the plane π parallel to the lines AB and CD. Calculate the ratio of the volumes of the parts if the ratio between the distances of the plane π from AB and CD is equal to k. *(Czechoslovakia)*

Second Day

4. Find four real numbers x_1, x_2, x_3, x_4 such that the sum of any of the numbers and the product of other three is equal to 2. *(Soviet Union)*

5. Given a triangle OAB such that $\angle AOB = \alpha < 90^\circ$, let M be an arbitrary point of the triangle different from O. Denote by P and Q the feet of the perpendiculars from M to OA and OB, respectively. Let H be the orthocenter of the triangle OPQ. Find the locus of points H when:
   (a) M belongs to the segment AB;
   (b) M belongs to the interior of $\triangle OAB$. *(Romania)*

6. We are given $n \ge 3$ points in the plane. Let d be the maximal distance between two of the given points. Prove that the number of pairs of points whose distance is equal to d is less than or equal to n. *(Poland)*

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com