1-st Hong Kong (China) Mathematical Olympiad 1998

December 29, 1998

1. Let $PQRS$ be a cyclic quadrilateral with $\angle PSR = 90^\circ$ and let H, K be the feet of the perpendiculars from Q to PR and PS respectively. Show that HK bisects QS.

2. The base of a pyramid is a convex 9-gon. Each diagonal of the base and each edge on the lateral surface of the pyramid is colored black or white (the sides of the base are not colored). Both colors are used. Prove that there are three segments of the same color which form a triangle.

3. Let be given nonzero integers s, t, and let (x, y) be any pair of integers. A move changes (x, y) to $(x+t, y-s)$. The pair (x, y) is called *good* if after several moves (possibly none) it becomes a pair of integers that are not coprime.

   (a) Is (s,t) a good pair?

   (b) Show that for any s, t there is a pair (x, y) which is not good.

4. Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a function with the following properties:

   (i) $f(1) = 1$,

   (ii) $f(x+1) = xf(x)$, and

   (iii) $f(x) = 10^{g(x)}$ for some convex function $g : \mathbb{R}^+ \to \mathbb{R}$.

   Prove that:

   (a) $t(g(n) - g(n-1)) \leq g(n+t) - g(n) \leq t(g(n+1) - g(n))$ for any integer $n > 1$ and any t with $0 \leq t \leq 1$;

   (b) $4/3 \leq f(1/2) \leq 4\sqrt{2}/3$.

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The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com