9-th Hong Kong (China) Mathematical Olympiad 2006

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1. A subset M of $\{1,2,\ldots,2006\}$ has the property that for any three elements x,y,z of M with $x<y<z$, $x+y$ does not divide z. Determine the largest possible size of M.

2. For a positive integer k, let $f_1(k)$ be the square of the sum of the digits of k. Define $f_{n+1}=f_1\circ f_n$. Evaluate $f_{2007}(2^{2006})$.

3. A convex quadrilateral $ABCD$ with $AB\neq CD$ is inscribed in a circle with center O. The diagonals AC and BD intersect at E. If P is a point inside $ABCD$ such that
$$\angle PAB+\angle PCB=\angle PBC+\angle PDC=90^\circ,$$
prove that O,P and E are collinear.

4. Let $(a_n)_{n\geq 1}$ be a sequence of positive numbers. If there is a constant $M>0$ such that $a_1^2+a_2^2+\cdots+a_n^2<Ma_{n+1}^2$ for all n, then prove that there is a constant $M'>0$ such that $a_1+a_2+\cdots+a_n<M'a_{n+1}$.

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